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DECOMPOSITION FOR VECTOR MEASURE

by

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0. Introduction

Let S be a set, $\sum$ a σ-ring of subsets of S, X a normed space, $m : \sum \to X$ a vector measure and μ a nonnegative measure on $\sum$. In [9] Sachie Ohba has proved that there exist unique m_0 and m_1 such that $m=m_0+m_1$, where $m_0 \ll \mu$ and $m_1 \perp \mu$. In §1 of this paper we shall prove that this result is valid for outer measure (or, signed measure). Let S be a locally compact Hausdorff space, $\mathcal{B}_w$ the σ-ring generated by closed sets of S. In [6] N.Y. Luther has proved that any nonnegative weakely Borel measure ν on $\mathcal{B}_w$ is uniquely decomposed by regular weakely Borel measure ν_1 and anti-regular weakely Borel measure ν_2. In § 2 of this paper we shall extend this result to the case of vector measure. And some properties of vector measure has proved.

1. The Lebesgue Decomposition Theorem

Let S be a set, $\sum$ a σ-ring of subsets of S, X a normed space. A set function m defined on $\sum$ with values in X is called a vector measure if for every sequence $\{E_n\}$ of mutually disjoint sets of we have $m(\bigcup_{n=1}^{\infty} E_n)=\sum_{n=1}^{\infty} m(E_n)$. Following [2] we say that a set function β defined on $\sum$ is an outer measure if it is nonnegative, countably subadditive, and it vanishes on the empty set ϕ. The set function λ defined on $\sum$ is called signed measure if it is extended real valued, countably additive.

[Definition 1] A vector measure m is continuous with respect to an outer measure (or signed measure) $\beta (m \ll \beta)$ if $\beta(E)=0$ implies $m(E)=0$ for every $E \in \sum$.

Following [4], let m be a vector measure on $\sum$. $\|m\|$ is called semi-variation of m if $\|m(E)\| =\sup|\sum_{i=1}^{n} d_i \; m(E_i)|$ for every $E \in \sum$, where superemum is taken over all finite collections of scalars with $|\alpha_i|$ and all partition of E into finite number of disjoint sets in $\sum$.

[Proposition 1] Let m be a vector measure on $\sum$. $m \ll \beta$ if and only if for every number $\varepsilon>0$ there exists a number $\delta=\delta(\varepsilon)>0$ such that for every $A \in \sum$ with $|\beta(A)|<\delta$ we have $\|m(A)\|<\varepsilon$.

Proof. Sufficiency is clear.

Necessity. (i) Let β is an outer measure. If it is false, then there exists a number $\varepsilon_0>0$ such that for every number $\delta>0$ there exists a set $A_\delta \in \sum$ such that $\beta(A_\delta)<\delta$ and $\|m(A_\delta)\| \geq \varepsilon_0$. Taking $\delta=1/2^n$ and $A_n=A_\delta$, we have $\beta(A_n)<1/2^n$ and $\|m(A_n)\| \geq \varepsilon_0$ for all n. If we put $B_n=\bigcup_{k=n}^{\infty} A_k$ and $B=\bigcap_{n=1}^{\infty} B_n$, then $\beta(B) \leq \beta(\bigcup_{k=n}^{\infty} A_k) \leq \sum_{k=n}^{\infty} \beta(A_k) \; 1/2^{n-1}$ for all n. Hence $\beta(B)=0$. For every $D \in \sum$ with $D \subset B$ $\beta(D)=0$. Since $m \ll \beta, m(D)=0$. We put $\widetilde{m}(E)=\sup\{\|m(A)\| : A \subset E, A \in \sum\}$ Then $\widetilde{m}(B) \neq 0$. On the other hand, $\widetilde{m}(B_n) \geq m(A_n) \geq \|m(A_n)\| \geq \varepsilon_0$ for all n. Since $\{B_n\}$ is decreasing sequence, by Gould ([5] corollary 3.6) we have $\widetilde{m}(B)=\lim_{n\to\infty} \widetilde{m}(B_n) \geq \varepsilon_0$. Therefore we get a contradiction.

(ii) Let β is a signed measure. By Berberian ([2] Theorem 50.1) $\beta \ll |\beta|$, where $|\beta|$ is total variation of β. Hence $m \ll |\beta|$. Therefore, proof is clear by above.

[Proposition 2] For any vector measure $m : \Sigma \longrightarrow X$ there exists a finite nonnegative measure ν on Σ such that:

(1) $m \ll \nu$, and

(2) $\nu(E) \leq \bar{m}(E) = \sup \{ \| m(A) \| : A \subset E, A \in \Sigma \}$ for every $E \in \Sigma$.

Proof. See Dinculeanu and Kluvanek ([3[Theorem 1).

[Definition 2] A vector measure m is singular with respect to an outer measure (or. signed measure) $\beta (m \perp \beta)$ if there exists a set $F \in \Sigma$ such that $E-F \in \Sigma$, $\beta(F)=0$ and $m(E-F)=0$ for every $E \in \Sigma$.

[Lemma 1] Any vector measure $m : \Sigma \longrightarrow X$ which both $m \ll \beta$ and $m \perp \beta$ is zero measure.

Proof. (i) Let β is an outer measure. Since $m \perp \beta$, there exists a set $F \in \Sigma$ such that $E-F \in \Sigma$, $\beta(F)=0$ and $m(E-F)=0$ for every $E \in \Sigma$. Since $m \ll \beta$, $m(E \cap F)=0$. Hence $m(E)=m(E \cap F)+m(E-F)=0$.

(ii) Let β is a signed measure. Since $m \perp \beta$, there exists a set $F \in \Sigma$ such that $E-F \in \Sigma$, $\beta(F)=0$ and $m(E-F)=0$ for every $E \in \Sigma$. Since $m \ll \beta$, $m(F)=0$. By Dunford and Schwartz ([4] Lemma IV 10.4) $\| m(F) \| =0$ and $0 \leq \| m(E \cap F \| \leq \| m(F) \| =0$. Hence $m(E \cap F)=0$. Therefore $m=0$.

[Theorem 1] Let $m : \Sigma \longrightarrow X$ be a vector measure and β an outer measure (or. signed measure) on Σ. Then there exist unique m_0 and m_1 such that m_0+m_1, where $m_0 \ll \beta$ and $m_1 \perp \beta$.

Proof. First, we suppose that β is an outer measure. By Proposition 2 there exists a finite nonnegative measure ν on Σ such that $m \ll \nu$. By Brooks ([1]) there exist unique ν_0 and ν_1 such that $\nu = \nu_0 + \nu_1$, where $\nu_0 \ll \beta$ and $\nu_1 \perp \beta$. Since $\nu_1 \perp \beta$, there exists a set $F \in \Sigma$ such that $E-F \in \Sigma$, $\beta(F)=0$ and $\nu_1(E-F)=0$ for every $E \in \Sigma$. Since $\nu_0 \ll \beta$, $\nu_0(E \cap F)=0$. Hence

$$\begin{aligned} \nu(E \cap F) &= \nu_0(E \cap F) + \nu_1(E \cap F) = \nu_1(E \cap F) \\ &= \nu_1(E \cap F) + \nu_1(E-F) = \nu_1(E) \\ \nu(E-F) &= \nu_0(E-F) + \nu_1(E-F) = \nu_0(E-F) \\ &= \nu_0(E-F) + \nu_0(E \cap F) = \nu_0(E). \end{aligned}$$

We put $m_0(E)=m(E-F)$ and $m_1(E)=m(E \cap F)$ for every $E \in \Sigma$. Then $m(E)=m(E-F)+m(E \cap F)=m_0(E)+m_1(E)$. If $\beta(E)=0$, then $\nu(E-F)=\nu_0(E)=0$. Since $m \ll \nu$, $m_0(E)=m(E-F)=0$. Hence $m_0 \ll \beta$. Since $\nu_1 \perp \beta$, there exists a set $F \in \Sigma$ such that $E-F \in \Sigma$, $\beta(F)=0$ and $\nu_1(E-F)=0$ for every $E \in \Sigma$. Hence $m_1(E-F)=m(E-F \cap F)=m(\phi)=0$. Therefore $m_1 \perp \beta$.

Let m_2 and m_3 be another decomposition for m. That is, $m=m_0+m_1=m_2+m_3$, where $m_2 \ll \beta$ and $m_3 \perp \beta$. Then $m_0-m_2=m_3-m_1$. Since $m_0 \ll \beta$ and $m_2 \ll \beta$, $(m_0-m_2) \ll \beta$. Since $m_1 \perp \beta$ and $m_3 \perp \beta$, there exist $F_1 \in \Sigma$, $F_2 \in \Sigma$ such that $E-F_1 \in \Sigma$, $\beta(F_1)=0$, $m_1(E-F_1)=0$, $E-F_2 \in \Sigma$, $\beta(F_2)=0$ and $m_3(E-F_2)=0$ for every $E \in \Sigma$. We put $F=F_1 \cup F_2$, then $\beta(F) \leq \beta(F_1) + \beta(F_2)=0$. Hence $\beta(F)=0$. By Dunford and Schwartz ([4] Lemma IV. 10.4) $\| m_1(E-F_1) \| =0$ and $\| m_3(E-F_2) \| =0$. Since $0 \leq \| m_1(E-F) \| \leq \| m_1(E-F_1) \| =0$, $m_1(E-F)=0$. Similarly $m_3(E-F)=0$. Therefore $(m_3-m_1) \perp \beta$. Since $(m_0-m_2) \ll \beta$ and $(m_0-m_2) \perp \beta$, $m_0=m_2$ and $m_1=m_3$ by Lemma 1

Next, if β is a signed measure. Since total variation $|\beta|$ is a measure, there exist unique m_0 and m_1 such that $m=m_0+m_1$, where $m_0 \ll |\beta|$ and $m_1 \perp |\beta|$. By Berberian ([2] Theorem 50.1) $\beta \ll |\beta|$. Hence $m_0 \ll \beta$ and $m_1 \perp \beta$.

2. The Decomposition Theorem of weakly Borel measure

Let S be a locally compact Hausdorff space, $\mathcal{B}(S)$ (resp. $\mathcal{B}_w$) is σ-ring generated by compact (resp. closed) sets of S, X a normed space. A set function m defined on $\mathcal{B}(S)$ (reesp. $\mathcal{B}_w$) with values in

X is called Borel vector measure (resp. weakely Borel vector measure; w.B. vector measure) if for every sequence $\{E_n\}$ is mutually disjoint sets of $\mathfrak{B}(S)$ $(\mathfrak{B}_w)$ we have $m(\bigcup_{n=1}^{\infty} E_n)=\sum_{n=1}^{\infty} m(E_n)$. A Borel vector measure $m: \mathfrak{B}(S) \longrightarrow X$ is called regular if for every $A \in \mathfrak{B}(S)$ and every $\varepsilon>0$ there exists a compact set $K \subset A$, $K \in \mathfrak{B}(S)$ and an open set $G \supset A$, $G \in \mathfrak{B}(S)$ such that for every $D \in \mathfrak{B}(S)$ with $D \subset G-K$ we have $\|m(D)\|<\varepsilon$. The w.B. vector measure $m: \mathfrak{B}_w \rightarrow X$ is called regular if for every $E \in \mathfrak{B}_w$ and every number $\varepsilon>0$ there exists a compact set $K \subset E, K \in \mathfrak{B}_w$ such that $\|m(E-K)\|<\varepsilon$. The Borel vector measure (or. w.B. vector measure) $m: \mathfrak{B}(S)$ (or. $\mathfrak{B}_w$) $\longrightarrow X$ is antiregular if $m \perp \mu$ for every non-negative regular Borel measure (or. regular w.B. measure) μ.

[Proposition 3] Any Borel vector measure $m: \mathfrak{B}(S) \rightarrow X$ is antiregular if and only if for every set $E \in \mathfrak{B}(S)$ there exists a set A such that $A \cap E \in \mathfrak{B}(S)$, $m(E-A)=0$ and such that $m(C)=0$ for every compact set $C \subset A$.

Proof. Sufficiency. Sachio Ohba has proved ([9] Proposition 6) Necessity. By Sachio Ohba ([9] Theorem 3) there exist unique regular m_0 and antiregular m_1 such that $m=m_0+m_1$, so that it suffices to show that $m_0=0$. Suppose that $m_0(E) \neq 0$ for some $E \in \mathfrak{B}(S)$. Then $m_0(E \cap A) \neq 0$ and by regularity of m_0 there exists a compact $C \subset E \cap A$ such that $m_0(C) \neq 0$ in this case, and that is contrary to hypothesis since $C \subset A$. Hence $m_0(E)=0$.

[Theorrm 2] Let m be a w.B. vector measure. Then there exist a unique regular w.B. vector measure m_0 such that:

(i) $\|m_0\| \leq \|m\|$, and

(ii) $\|n\| \leq \|m_0\|$ for every regular w.B. vector measure which satisfies $\|n\| \leq \|m\|$.

Proof. By [8] there exist unique regular w.B. vector measure m_0 and anti-regular w.B. vector measure m_1 such that $m=m_0+m_1$. Let n be a regular w.B. vector measure such that $\|n\| \leq \|m\|$. By Proposition 3 of [8] $n \perp m_1$. Then there exists a set $F \in \mathfrak{B}_w$ such that $E-F \in \mathfrak{B}_w, m_1(F)=0$ and $n(E-F)=0$ for every $E \in \mathfrak{B}_w$. Hence $n(E)=n(E \cap F)$ for every $E \in \mathfrak{B}_w$. Since $\|n\| \leq \|m_0\|+\|m_1\|$ and $\|m_1(E \cap F)\|=0$, $\|n(E)\|=\|n(E \cap F)\| \leq \|m_0(E \cap F)\|+\|m_1(E \cap F)\|=\|m_0(E \cap F)\| \leq \|m_0(E)\|$ for every $E \in \mathfrak{B}_w$. Since $\|m_0(E)\|=\|m(E \cap F)\|$, $\|m_0(E)\| \leq \|m(E)\|$ for every $E \in \mathfrak{B}_w$, this completes proof.

[Definition 3] A w.B. vector measure $m: \mathfrak{B}_w \rightarrow X$ is weakely antiregular if largestest regular w.B. vector measure (denote $\bar{m}$) in the sence of Theorem 2 has zero semi-bariation.

By definition m is both weakely anti-regular and regular if and only if, $m=0$.

[Lemma 2] Let m be a w.B. vector measure: $\mathfrak{B}_w \rightarrow X$. If m is antiregular, then m is weakely antiregular.

Proof. Since $\bar{m}$ is regular, $\bar{m} \perp m$. Then there exists a set $F \in \mathfrak{B}_w$ such that $E-F \in \mathfrak{B}_w$, $m(F)=0$ and $\bar{m}(E-F)=0$ for every $E \in \mathfrak{B}_w$. Since $\|\bar{m}\| \leq \|m\|$, $\|\bar{m}(F)\|=0$. Hence $\|\bar{m}(E \cap F)\|=0$. Therefore $\|\bar{m}(E)\|=0$ for every $E \in \mathfrak{B}_w$. Consequentely m is weakely anti-regular.

[Theorem3] Let m be a w.B. vector measure; $\mathfrak{B}_w \rightarrow X$. m is weakely anti-regular if and only if $n \perp m$ for every regular w.B. vector measure n.

Proof. Sufficiency. Since $\bar{m} \perp m$, there exists a set $F \in \mathfrak{B}_w$ such that $E-F \in \mathfrak{B}_w$, $m(F)=0$ and $\bar{m}(E-F)=0$ for every $E \in \mathfrak{B}_w$. Since $\|\bar{m}\| \leq \|m\|$, $\|\bar{m}(E \cap F)\| \leq m(E \cap F)\| \leq \|m(F)\|=0$. Hence $\bar{m}(E \cap F)=0$. Therefore m is weakely anti-regular.

Necessity. Let n be a arbitrary regular w. B. vector measure, By Proposition 2 there exist nonnegative measure ν, μ suchthat $m \ll \mu$ and $n \ll \nu$. Since ν is regular and μ is antiregular ([8] Theorem 2 and

Theorem 5), $\nu \perp \mu$ by N.Y. Luther ([7] Theorem 2). Since $m < \mu$ and $n < \nu$, $n \perp m$.

[Theorem 4] Let m be a w.B. vector measure : $\mathcal{B}_w \rightarrow X$. Then there exist unique regular w.B. vector measure m_0 and weakely anti-regular w.B. vector measure m_1 such that $m = m_0 + m_1$.

Proof. It is clear by [8] and Lemma 2.

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