A note on H-closed spaces

by

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1. INTRODUCTION.

In this paper I want to find out the properties which is satisfied in H-closed space. Many characterizations was introduced by Larry L. Herrington and Paul E. Long with weakly continuous mapping [4] and strongly closed graph, filterbase, family of regular-closed sets and net [5].

Here I calculate the property that a continuous surjection image of H-closed space is H-closed, a preimage of continuous bijection of H-closed space and Urysohn space is H-closed space, a H-closed subspace of Hausdorff space is closed and if product space is H-closed space then each projection space is H-closed but not for converse.

Moreover all the notation are based on [1].

2. PRELIMINARY AND DEFINITION.

Definition 2.1. A Hausdorff space $X$ is H-closed if for every $U_{\alpha} | \alpha \in \Delta$ there exists a finite subfamily $\{U_{\alpha_{i}}\} i=1,2, \ldots, n$ such that $\bigcup_{i=1}^{n} \text{Cl}(U_{\alpha_{i}}) = X$.

Definition 2.2. A mapping $f : X \rightarrow Y$ is said to be weakly continuous (briefly w.c.) if for each point $x \in X$ and each open set $V \subseteq Y$ containing $f(x)$, there exists an open set $U \subseteq X$ containing $x$ such that $f(U) \subseteq \text{Cl}(V)$.

We omit the other definition.

Lemma 2.3. (Levine). A mapping $f : X \rightarrow Y$ is w.c. iff for each open set $V \subseteq Y$, $f^{-1}(V) \subseteq \text{Int}(f^{-1}(\text{Cl}(V)))$.

Proof. [3].

Lemma 2.4 If $Y$ is a Urysohn space and $f : X \rightarrow Y$ is w.c. injection, then $X$ is Hausdorff.

Proof. For any distinct points $x_{1}, x_{2} \in X$, we have $f(x_{1}) \neq f(x_{2})$ because $f$ is injective. Since $Y$ is Urysohn, there exist open sets $V_{1}$ and $V_{2}$ in $Y$ such that $f(x_{1}) \not\in V_{1}$, $f(x_{2}) \not\in V_{2}$ and $\text{Cl}(V_{1}) \cap \text{Cl}(V_{2}) = \emptyset$. Hence we have $\text{Int}(f^{-1}(\text{Cl}(V_{1}))) \cap \text{Int}(f^{-1}(\text{Cl}(V_{2}))) = \emptyset$. Since $f$ is w.c., by L. 2.3., we have $x_{1} \not\in f^{-1}(V_{j}) \subseteq \text{Int}(f^{-1}(\text{Cl}(V_{j})))$ for $j=1,2$. This implies that $X$ is Hausdorff.

3. MAIN PROPERTIES.

Theorem 3.1. A continuous surjection image of H-closed space is H-closed.

Proof. Let $f : X \rightarrow Y$ be a continuous surjection and $X$ be a H-closed space, $\{U_{\alpha} | \alpha \in \Delta \}$ be arbitrary open cover of $Y$ then there exists open cover of $X$ with $\{f^{-1}(U_{\alpha}) | \alpha \in \Delta \}$ since $f$ is continuous. On the other hand since $X$ is H-closed space there exists finite subcover $\{f^{-1}(U_{\alpha_{i}}) | i=1,2, \ldots, n \}$ of $\{f^{-1}(U_{\alpha}) | \alpha \in \Delta \}$ such that $\bigcup_{i=1}^{n} \text{Cl}(f^{-1}(U_{\alpha_{i}})) = Y$. Thus, for above finite subcover, $\{U_{\alpha_{1}}, U_{\alpha_{2}}, \ldots, U_{\alpha_{n}}\}$ is finite subcover of $\{U_{\alpha} | \alpha \in \Delta \}$ and moreover $Y = f(X) = f \left( \bigcup_{i=1}^{n} \text{Cl}(f^{-1}(U_{\alpha_{i}})) \right)$.
\( \bigcap_{i=1}^{n} \bigcup_{a \in A} (\text{Cl}(U_{a})) = Y \). Clearly \( \bigcup_{a \in A} (\text{Cl}(U_{a})) \subseteq Y \). Thus \( \bigcup_{a \in A} (\text{Cl}(U_{a})) = Y \). This implies \( Y \) is H-closed space.

Theorem 3.2. A preimage of continuous bijection of H-closed and Urysohn space is H-closed space.

Proof. We put \( f : X \to Y \) be continuous bijection and \( Y \) is H-closed and Urysohn space, then by Lemma 2.4 \( X \) is Hausdorff generally since \( f \) is continuous then \( f \) is w.c. [4].

If each open cover \( \{U_a : a \in A\} \) of \( X \) is the preimage of some open cover of \( Y \) of the form \( \{V_a : a \in A\} \) Thus \( \{f^{-1}(V_a) : a \in A\} \) be a cover of \( X \) such that \( f^{-1}(V_a) = U_a \) for each \( a \in A \). Since \( Y \) is H-closed there exist finite subcover \( \{V_{a_i} : i = 1, 2, \ldots, n\} \) of \( \{V_a : a \in A\} \) such that \( \bigcup_{i=1}^{n} \text{Cl}(V_{a_i}) = Y \). For this finite subcover \( \{f^{-1}(V_{a_i}) : i = 1, 2, \ldots, n\} \) is the finite subcover of \( \{f^{-1}(V_a) : a \in A\} \) and that \( \bigcup_{i=1}^{n} \text{Cl}(f^{-1}(V_{a_i})) = f^{-1}(\bigcup_{i=1}^{n} \text{Cl}(V_{a_i})) = f^{-1}(Y) = X \). Since \( f \) is surjection, Suppose if \( x \in X \), \( x \in \bigcup_{i=1}^{n} \text{Cl}(f^{-1}(V_{a_i})) \), then \( x \in f^{-1}(\bigcup_{i=1}^{n} \text{Cl}(V_{a_i})) = f^{-1}(\bigcup_{i=1}^{n} \text{Cl}(V_{a_i})) = f^{-1}(Y) = X \) since \( f \) is surjection.

\( f(x) \in Y \) Thus this implies \( X \) is H-closed.

Theorem 3.3. A H-closed subspace of Hausdorff space is closed.

Proof. Let \( X \) be Hausdorff space and \( A \subseteq X \) be the H-closed subspace, then as for the relative topology, we pick any open covering \( \{U(a) \cap A : a \in A\} \) of \( A \). On the other hand, since \( U(a) \) is open in Hausdorff space and \( A \) is H-closed space itself, there exists finite subcover \( \{U(a_i) \cap A : i = 1, 2, \ldots, n\} \) of \( \{U(a) \cap A : a \in A\} \) such that \( \bigcup_{i=1}^{n} \text{Cl}(U(a_i) \cap A) = A \). Since finite union of closed sets is closed, thus \( A \) is closed.

Theorem 3.4. Let \( \{Y_a : a \in A\} \) be a family of spaces. If \( \bigcup_a Y_a \) is H-closed space, then \( Y_a \) is H-closed for each \( a \in A \).

Proof. We take projection \( p_a : \bigcup_a Y_a \to Y_a \), since projection is continuous surjection and by theorem 3.1 \( Y_a \) is H-closed space.

In above theorem 3.4, converse is not hold by property of theorem 3.2.

References

(2) N. Bourbaki, General topology, Part 1, Addison-Wesley, Reading, Mass., 1966

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