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On the eigenvectors of k_{ij} in X_2

by

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I. INTRODUCTION

Let X_2 be a two-dimensional space referred to a real coordinate system x^i and endowed with a real quadratic tensor g_{ij}, which may be split into its symmetric part h_{ij} and its skew-symmetric part k_{ij}:

(1.1) $$g_{ij}=h_{ij}+k_{ij}.$$

In the present paper, all coordinate transformations $\bar{x}^i$ x^i are assumed to satisfy

$$\left|\frac{\partial \overline{X}}{\partial X}\right| \neq 0.$$

Furthermore, we assume that

(1.2) $$g \overset{\text{def}}{=\!=} |g_{ij}| \neq 0, \quad h \overset{\text{def}}{=\!=} |h_{ij}| \neq 0. \quad k \overset{\text{def}}{=\!=} |k_{ij}| \neq 0,$$

According to (1.2) there exists a unique symmetric tensor h^{ij} defined by

(1.3) $$h_{ij}h^{ik}=\delta_j^k.$$

Therefore we may use both h_{ij} and h^{ij}as the tensors for raising and/or lowering indices of all tensors defined in X_2 in the usual manner.

The purpose of the present paper is to study the properties of the eigenvalues M and the corresponding eigenvectors e^i in X_2, defined by

(1.4) $$(Mh_{ij}+k_{ij})e^j=0,$$

where M is an arbitrary scalar.

II. EIGENVALUES OF k_{ij}.

In the present section we shall find the eigenvalues M of (1.4) for two possible cases in X_2.

LEMMA (2.1). *We have*

(2.1) $$k>0.$$

Proof. Since k_{ij}, is skew-symmetric, our assertion follows from (1.2) and from

$$k=(k_{12})^2.$$

LEMMA (2.2). *We have*

(2.2) $$A(M) \overset{\text{def}}{=\!=} \mathrm{Det}((Mh_{ij}+k_{ij}))=hM^2+k.$$

Proof. This is a direct result of (3.3)b of [1], when $n=2$.

THEOREM 2.3. *The eigenvalues M_1 and M_2 of* (1.4) *are given by* (For the case $h>0$)

(2.3)a $$M_1=-M_2=i\sqrt{\bar{k}}, \text{ where } \bar{k} \overset{\text{def}}{=\!=} k/h,$$

(For the case $h<0$)

(2.3)b $$M_1=-M_2=\sqrt{-\bar{k}}$$

Proof. The existence of the above two cases is clear from the Lemma (2.1). A necessary and sufficient condition for the existence of a non-trivial solution e^i of (1.4) is $A(M)=0$. According to the Lemma (2.2) this condition is equivalent to

$$M^2+\bar{k}=0,$$

which admits the solutions M_1 and M_2 given by (2.3).

III. EIGENVECTORS OF k_{ij}.

In this section we shall study and derive several properties of eigenvectors e^i given by (1.4) in X_2.

Theorem (3.1). *The eigenvector e^i is a null vector:*

(3.1) $$h_{ij}e^ie^j=0.$$

Proof. Since k_{ij} is skew-symmetric, we have according to (1.4)

(3.2) $$Mh_{ij}e^ie^j=-k_{ij}e^ie^j=0.$$

Hence our assertion holds since M is found to be non-zero in the Theorem (2.3).

Theorem (3.2). *The eigenvector e^i is also an eigenvector of each of the tensors ${}^{(p)}g_{ij}$ and ${}^{(p)}k_{ij}$, defined by*

(3.3) $${}^{(1)}g_{ij}\overset{\text{def}}{=\!=}g_{ij},\quad {}^{(2)}g_{ij}\overset{\text{def}}{=\!=}g_i{}^kg_{kj},\quad {}^{(p)}g_{ij}\overset{\text{def}}{=\!=}g_i{}^k{}_{(p-1)}g_{kj},\quad (p=1,2,3,\cdots\cdots).$$

Proof. First, we have according to (1.4)

$$g_{ij}e^j=(h_{ij}+k_{ij})e^j=(1-M)h_{ij}e^j,$$

and in a similar manner,

$${}^{(p)}g_{ij}e^j=(1-M)^ph_{ij}e^j.$$

Therefore, e^i is also an eigenvector of ${}^{(p)}g_{ij}$. The last assertion may be proved similarly.

Let $\underset{1}{e^i}$ and $\underset{2}{e^i}$ be the eigenvectors corresponding to M_1 and M_2 given in (2.3), respectively.

Theorem (3.3). *There are only two eigenvectors $\underset{1}{e^i}$ and $\underset{2}{e^i}$, and they have the following properties:*

(a) *They are defined up to arbitrary factors of proportionality.*

(b) *They are null vectors.*

(c) *They satisfy the conditions*

(3.4) $$h_{ij}\underset{1}{e^i}\underset{2}{e^j}\neq 0.$$

(d) *Their directions are complex-conjugate for the case $h>0$, and real for the case $h<0$.*

Proof. Since M_1 and M_2 are different from zero and distinct, (1.4) admits only two eigenvectors defined up to arbitrary factors of proportionality. Statement (b) follows from Theorem (3.1) and Theorem (2.3), and statement (c) follows from statement (b). Statement (c) is clear, since M_1 and M_2 are complex-conjugate for the case $h>0$ and real for the case $h<0$.

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〈국문초록〉 「X_2에서 K_{ij}의 eigenvector에 관하여」

崔 大 鎬

이 논문의 목적은 M이 임의의 scalar일때 $(Mh_{ij}+K_{ij})e^j=0$으로 정의된 2차원 공간 X_2에서의 eigenvalue M과 eigenvector e^i에 관한 여러가지 성질을 얻는데 있다.