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SOME REMARKS CONCERNING PERMUTATIONS ON SYMMETRY CLASSES

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1. Introduction

Let F be a field, G a subgroup of the symmetric group S_m, and $\chi: G\to F$ be a character of degree 1. Let $V_1, \cdots, V_m$ be finite dimensional vector spaces over F such that $V_i=V_{\sigma(i)}$ for $i=1,\cdots,m$ and for all $\sigma\in G$. Let W be any vector space over F. An m-multilinear mapping $f:\times_{i=1}^{m} V_i\to W$ is said to be *symmetric* with respect to G and χ if

$$f(x_{\sigma(1)},\cdots,x_{\sigma(m)})=\chi(\sigma)f(x_1,\cdots,x_m)$$

for any $\sigma\in G$ and arbitrary $x_i\in V_i$. A pair (P,μ) consisting of a vector space P over F and an m-multilinear function $\mu:\times_{i=1}^{m} V_i\to P$, symmetric with respect to G and χ, is a *symmetry class of tensors* over $V_1,\cdots,V_m$, associated with G and χ if the following universal factorization property is satisfied.

For any vector space U over F and any m-multilinear function $g:\times_{i=1}^{m} V_i\to U$, symmetric with respect to G and χ, there exists a unique linear mapping $h: P\to U$ such that $g=h\mu$.

The symmetry class of tensors associated with G and χ always exists and is unique up to vector space isomorphism (see [2], [4]). We denote such a space by $(V_1,\cdots,V_m)_\chi(G)$. When $V_1=\cdots=V_m=V$, it is usually denoted by $V_\chi^m(G)$ [2]. The decomposable element $\mu(x_1,\cdots,x_m)$, $x_i\in V_i$, $i=1,\cdots,m$, is denoted by $x_1*\cdots*x_m$.

Let $T_i: V_i\to V_i$ be linear mappings such that $T_i=T_{\sigma(i)}$ for $i=1,\cdots,m$ and all $\sigma\in G$. Then there exists a unique linear mapping $K(T_1,\cdots,T_m)$ on $(V_1,\cdots,V_m)_\chi(G)$ such that $K(T_1,\cdots,T_m)x_1*\cdots*x_m=T_1x_1*\cdots*T_mx_m$. If $T_1,\cdots,T_m$ are nonsingular and $(V_1,\cdots,V_m)_\chi(G)\neq\{0\}$ then clearly $K(T_1,\cdots,T_m)^{-1}=K(T_1^{-1},\cdots,T_m^{-1})$. If $T_1=\cdots=T_m=T$, $K(T_1,\cdots,T_m)$ is usually denoted by $K(T)$ [2].

Let $B_i=\{v_{i1},\cdots,v_{is_i}\}$ be bases of V_i, $1\leq i\leq m$, such that for each i,

$$v_{ij_i}=v_{\sigma(i)j_i}$$

for all $\sigma\in G$, $1\leq j_i\leq \dim V_i$. Let Γ denote the set of all m-tuples $\alpha=(\alpha_1,\cdots,\alpha_m)$ where α_i are positive integers such that $1\leq\alpha_i\leq\dim V_i$, $i=1,\cdots m$. If $\alpha\in\Gamma$, $\sigma\in G$, let $\alpha^\sigma=(\alpha_{\sigma(1)},\cdots,\alpha_{\sigma(m)})$. The group G induces an equivalence relation $\equiv$ on Γ as follows:

$$\alpha\equiv\beta \text{ if } \alpha^\sigma=\beta \text{ for some } \sigma\in G.$$

In each of the equivalence classes choose the m-tuple which is first in lexicographical order and let Δ denote the resulting system of distinct represeptatives. For each $\alpha\in\Delta$, let

$$G_\alpha=\{\sigma\in G:\alpha^\sigma=\alpha\}$$

and let

$$\bar{\Delta}=\{\alpha\in\Delta:\chi(\sigma)=1 \text{ for all } \sigma\in G_\alpha\}.$$

Let $v_\alpha{}^*=v_{1\alpha_1}{}^*\cdots{}^*v_{m\alpha_m}$, $\alpha\in\Gamma$. Then it can be shown that

$$B=\{v_\alpha{}^*:\alpha\in\bar{\Delta}\}$$

forms a basis of $(V_1,\cdots,V_m)_\chi(G)$ (see [1], [4]).

For each $\omega\in\bar{\Delta}$, let $\delta(\omega)$ denote the number of distinct integers in ω. Let $p=\min\{\delta(\omega):\omega\in\bar{\Delta}\}$.

Let U be a finite dimensional vector space over F. A linear mapping $g:U\to U$ is called a *generalized permutation* w.r.t. the basis $u_1,\cdots,u_n$ of U if $g(u_i)=c_iu_{\phi(i)}$ for some $\phi\in S_n$ and some non-zero scalars c_i. g is called a *permutation* w.r.t. the basis $u_1,\cdots,u_n$ if $c_i=1$ for all i.

Throughout this note we assume $(V_1,\cdots,V_m)_\chi(G)\neq\{0\}$ and let O_i be the orbit of G to which i belongs. Our purpose is to prove the following generalization of Theorem 3 in [2].

THEOREM. *Assume that* $\dim V_i>\min\{|O_i|,p\}$ *for* $i=1,\cdots,m$ *or* $\chi\equiv1$. *Then* $K(T_1,\cdots,T_m)$ *on* $(V_1,\cdots,V_m)_\chi(G)$ *is a generalized permutation w.r.t. the basis* B *if and only if* T_i *is a generalized permutation w.r.t. the basis* B_i *for each* i.

COROLLARY. *Suppose that* $\chi\equiv1$. *Then* $K(T_1,\cdots,T_m)$ *on* $(V_1,\cdots,V_m)_\chi(G)$ *is a permutation w.r.t. the basis* B *if and only if* $T_i=\lambda_iP_i$ *where* P_i *is a permutation w.r.t. the basis* B_i *for each* i *and* $\prod_{i=1}^m\lambda_i=1$.

Let $v_1,\cdots,v_n$ be an orthonormal basis of a unitary space V. In [2] Marcus and Minc proved that if $\chi\equiv1$ and rank $T>m$, then $K(T)$ is a permutation on

$V_\chi^m(G)$ w.r.t. the orthonormal basis $\left\{\left(\frac{|G|}{|G_\alpha|}\right)^{\frac{1}{2}} v_{\alpha_1} * \cdots * v_{\alpha_m} : \alpha \in \bar{\Delta}\right\}$ implies that $T=\lambda P$ where P is a permutation on V w.r.t. the orthonormal basis $v_1, \cdots, v_n$ and $\lambda^m=1$. We remark that the hypothesis rank $T > m$ can be dropped.

2. Proof of the theorem

We first prove following generalization of the lemma in [2]

LEMMA 1. *If* $x_1 * \cdots * x_m = y_1 * \cdots * y_m \neq 0$, *then for each orbit* O *of* G, $\{x_i : i \in O\}$ *and* $\{y_i : i \in O\}$ *span the same subspace.*

PROOF. Suppose for some $j \in O$, $x_j \notin \langle y_i : i \in O \rangle$, the subspace spanned by $\{y_i : i \in O\}$. Let $T_j : V_j \to V_j$ be a linear mapping such that

$$T_j(x_j)=0, \ T_j | \langle x_i : i \in O \rangle = \text{identity mapping.}$$

If $k \in O$, let $T_k = T_j$. If $k \notin O$, let T_k be the identity mapping on V_k. Then

$$K(T_1, \cdots, T_m) x_1 * \cdots * x_m = K(T_1, \cdots, T_m) y_1 * \cdots * y_m.$$

This implies that $0 = y_1 * \cdots * y_m$, a contraction. Therefore for any $j \in O$, $x_j \in \langle y_i : i \in O \rangle$. Similarly

$$\langle y_i : i \in O \rangle \subset \langle x_i : i \in O \rangle.$$

Hence $\langle y_i : i \in O \rangle = \langle x_i : i \in O \rangle$ and the lemma is proved.

LEMMA 2. *Let* $\omega \in \Delta$ *such that* $v_\omega^* \neq 0$. *Let* η_i *be permutations on* $\{1, \cdots, \dim V_i\}$ *such that* $\eta_i = \eta_{\sigma(i)}$ *for* $i=1, \cdots, m$ *and for all* $\sigma \in G$. *Then* $(\eta_1(\omega_1), \cdots, \eta_m(\omega_m)) \equiv \gamma$ *for some* $\gamma \in \bar{\Delta}$.

PROOF. Let $\eta(\omega) = (\eta_1(\omega_1), \cdots, \eta_m(\omega_m))$. By the hypothesis on η_i, we see that there are nonsingular linear mappings f_i on V_i such that $f_i = f_{\sigma(i)}$ for $i=1, \cdots, m$ and all $\sigma \in G$ and

$$f_i(v_{i\omega_i}) = v_{i\eta_i(\omega_i)}, \ i=1, \cdots, m.$$

Since $K(f_1, \cdots, f_m)$ is nonsingular, it follows that

$$K(f_1, \cdots, f_m) v_\omega^* = v_{\eta(\omega)}^* \neq 0.$$

If $\eta(\omega) \equiv \alpha$ for some $\alpha \in \Delta \backslash \bar{\Delta}$ then by Lemma 6.1 [4] $v_{\eta(\omega)}^* = 0$, a contradiction. Hence $\eta(\omega) \equiv \gamma$ for some $\gamma \in \bar{\Delta}$.

LEMMA 3. *If* $K(T_1, \cdots, T_m)$ *is nonsingular then* $T_i : V_i \to V_i$ *is nonsingular for* $i=1, \cdots, m$.

PROOF. Suppose $T_1(u_{11})=0$ for some nonzero vector u_{11} in V_1. For each $i=1, \cdots,$ m, let $D_i = \{u_{i1}, \cdots, u_{is_i}\}$ be a basis of V_i such that $u_{\sigma(i)j_i} = u_{ij_i}$ for all $\sigma \in G$ where

$1 \leq j_i \leq \dim V_i$. Since $(V_1, \cdots, V_m)_\chi(G) \neq \{0\}, \bar{\Delta} \neq \phi$. Let $\alpha \in \bar{\Delta}$. Then $u_\alpha{}^* \neq 0$. Let η_i be permutations on $\{1, \cdots, \dim V_i\}$ such that $\eta_i = \eta_{\sigma(i)}$ for all i and all $\sigma \in G$ and $\eta_1(\alpha_1) = 1$. By Lemma 2, $\eta(\alpha) = (\eta_1(\alpha_i), \cdots, \eta_m(\alpha_m)) \equiv \gamma$ for $\gamma \in \bar{\Delta}$ and hence $u_{\eta(\alpha)}{}^* \neq 0$. However,

$$K(T_1, \cdots, T_m) u_{\eta(\alpha)}{}^* = 0.$$

This contradicts the hypothesis that $K(T_1, \cdots, T_m)$ is nonsingular. Hence T_1 is nonsingular. Similarly T_i is nonsingular for $i \geq 2$.

PROOF of the theorem (Necessity) Case (i) : $\dim V_i > \min(|O_i|, p)$. Let θ be the permutation on $\bar{\Delta}$ such that for each $\alpha \in \bar{\Delta}$

$$K(T_1, \cdots, T_m) v_\alpha{}^* = \lambda_\alpha v_{\theta(\alpha)}{}^*$$

for some nonzero scalar λ_α. We shall show that for each $1 \leq j \leq \dim V_1$,

$$T_1(v_{1j}) = \lambda_{1j} v_{1\phi(j)}$$

for some positive integer $\phi(j)$. Let $\omega \in \bar{\Delta}$ such that $\delta(\omega) = p$. Then $v_\omega{}^* \neq 0$. Let $|\{\omega_i : i \in O_1\}| = k$. Then $k \leq \min(|O_1|, p) < \dim V_1$. For each $2 \leq t \leq k+1$ we are able to choose permutations η_i^t on $\{1, \cdots, \dim V_i\}$ such that

$$\text{(i) } \eta_i^t = \eta_{\sigma(i)}^t \text{ for all } i \text{ and all } \sigma \in G$$

and

$$\text{(ii) } \{\eta_i^t(\omega_i) : i \in O_1\} = (1, \cdots, \hat{t}, \cdots, k+1\}.$$

Let $\eta^t(\omega) = (\eta_1^t(\omega_1), \cdots, \eta_m^t(\omega_m))$. By Lemma 2, $\eta^t(\omega) \equiv \gamma^t$ for some $\gamma^t \in \bar{\Delta}$. Clearly $\{\eta_i^t(\omega_i)\ i \in O_1\} = \{\gamma_i^t : i \in O_1\}$. Since

$$K(T_1, \cdots, T_m) v_{\gamma^t}{}^* = \lambda_{\gamma^t} v_{\theta(\gamma^t)}{}^* \neq 0$$

it follows from Lemma 1 that

$$\langle T_i(v_{i\gamma_i^t}) : i \in O_1 \rangle = \langle v_{i\theta(\gamma^t)_i} : i \in O_1 \rangle.$$

This implies that

$$\langle T_1(v_{11}), \cdots, \widehat{T_1(v_{1t})}, \cdots, T_1(v_{1(k+1)}) \rangle = \langle v_{i\theta(\gamma^t)_i} : i \in O_1 \rangle.$$

Hence

$$\bigcap_{t=2}^{k+1} \langle T_1(v_{11}), \cdots, \widehat{T_1(v_{1t})}, \cdots, T_1(v_{1(k+1)}) \rangle = \bigcap_{t=2}^{k+1} \langle v_{i\theta(\gamma^t)_i} : i \in O_1 \rangle.$$

Since T_1 is nonsingular (Lemma 3), $T_1(v_{11}), \cdots, T_1(v_{1(k+1)})$ are linearly independent. Hence the left hand of the above equality is $\langle T_1(v_{11}) \rangle$. This shows that

$$\langle T_1(v_{11}) \rangle = \langle v_{1\phi(1)} \rangle,$$

for some integer $\phi(1)$. Similarly $\langle T_1(v_{1j}) \rangle = \langle v_{1\phi(j)} \rangle$ for some integer $\phi(j)$. Since T_1 is nonsingular, T_1 is a generalized permutation w.r.t. the basis B_1.

Similarly T_i is a generalized permutation w. r. t. the basis B_i for $i \geq 2$.

Case (ii) : $\chi \equiv 1$. Since $\chi \equiv 1$, for each $\gamma \in \Gamma$, $v_\gamma^* \in B$. For each $1 \leq t \leq \dim V_1$, let $\omega^t = (\omega_1^t, \cdots, \omega_m^t)$ such that $\omega_i^t = t$ if $i \in O_1$, $\omega_j^t = 1$ if $j \notin O_1$. Since $v_{\omega^t}^* \in B$, it follows that

$$K(T_1, \cdots, T_m)v_{\omega^t}^* = \lambda_{\omega^t} v_{\alpha^t}^*,$$

for some $\lambda_{\omega^t} \in F$ and some $\alpha^t \in \bar{\Delta}$. By Lemma 1,

$$\begin{aligned} \langle T_1(v_{1\omega_i^t}) : i \in O_1 \rangle &= \langle v_{1\alpha_i^t} : i \in O_1 \rangle \\ &= \langle T_1(v_{1t}) \rangle. \end{aligned}$$

Hence $\langle T_1(v_{1t}) \rangle = \langle v_{1\phi(t)} \rangle$ for some integer $\phi(t)$. This implies that T_1 is a generalized permutation w. r. t. the basis B_1. Similarly T_i is a generalized permutation w. r. t. the basis B_i for $i \geq 2$.

(Sufficiency). Suppose that for each $i = 1, \cdots, m$, there is a permutation θ_i on $\{1, \cdots, \dim V_i\}$ such that

$$T_i(v_{ij_i}) = \lambda_{ij_i} v_{i\theta_i}(j_i), \quad 1 \leq j_i \leq \dim V_i$$

for some nonzero scalars λ_{ij_i}. Then for $\omega \in \bar{\Delta}$,

$$K(T_1, \cdots, T_m)v_\omega^* = \prod_{i=1}^m \lambda_{i\omega_i} v_{1\theta_1(\omega_1)} * \cdots * v_{m\theta_m(\omega_m)} \neq 0.$$

Hence Lemma 2 implies that $(\theta_1(\omega_1), \cdots, \theta_m(\omega_m)) = \gamma^\sigma$ for some $\gamma \in \bar{\Delta}$ and some $\sigma \in G$. Hence

$$K(T_1, \cdots, T_m)v_\omega^* = \left(\prod_{i=1}^m \lambda_{i\omega_i}\right)\chi(\sigma)v_\gamma^*.$$

Since $K(T_1, \cdots, T_m)$ is nonsingular it is then clear that $K(T_1, \cdots, T_m)$ is a generalized permutation w. r. t. the basis B. This proves the sufficiency.

PROOF of the corollary. The sufficiency is trivial. We prove the necessity. In view of the theorem, for each i, there exists $\phi_i \in S_{\dim V_i}$ such that

$$T_i(v_{ij_i}) = c_{ij_i} v_{i\phi_i(j_i)}, \quad 1 \leq j_i \leq \dim V_i,$$

for some nonzero scalars c_{ij_i}. For each $1 \leq t \leq \dim V_1$,

$$K(T_1, \cdots, T_m)v_{1t} * v_{21} * \cdots * v_{m1} = \lambda_{1t}\lambda_{21} \cdots \lambda_{m1} v_{1\phi_1(t)} * v_{2\phi_2(1)} * \cdots * v_{m\phi_m(1)}.$$

Since $v_{1\phi_1(t)} * \cdots * v_{m\phi_m(1)} \in B$, it follows that

$$\lambda_{1t}\lambda_{21} \cdots \lambda_{m1} = 1.$$

Hence $\lambda_{11} = \lambda_{1t}$ for any $1 \leq t \leq \dim V_1$. This proves that $T_1 = \lambda_{11} P_1$ where P_1 is a permutation w. r. t. the basis B_1. Similarly we can show that $T_i = \lambda_i P_i$ for some scalar λ_i and some permutation P_i w. r. t. the basis B_i, $i \geq 2$. Clearly $\prod_{i=1}^m \lambda_i = 1$.

This completes the proof.

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