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DINI DERIVATES AND APPLICATIONS

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0. Introduction

In advanced treatments of differentiability of real-valued functions of a real variable the four Dini Derivates are introduced first and the questions of their finiteness and equality are discussed subsequently. It is natural that in cases where the underlying function is not necessarily differentiable the part of the derivative could, to some extent, be successfully played by the derivates. This, however, depends on knowing the extent to which the derivates resemble the derivative when dealing with functions individually and in combination. This note seeks to establish some elementary results in this context and concludes by pointing out some less elementary applications of this calculus of the derivates.

In section 1 the standard definitions of the Dini Derivates are recalled (mainly in order to fix the notation) and two preliminary lemmas are proved. Some rules for the derivates analogous to the rules of differentiation are obtained in section 2, the proofs being confined to pointing out the essential details. Zygmund's lemmas on the derivates and their consequences are quoted from [5] in section 3 in order to make the discussion somewhat self-contained. These results are stated in form and order different from those in [5] and although it is easy to prove them in the order in which they are presented here, it is preferred to omit the proofs. Finally three applications of the rules developed are provided in section 4.

1. Notations and preliminaries

NOTATION 1.1. Let a, b be real numbers with $a<b$. If $\varphi : (a,b] \to \mathbf{R}$ then $m(\varphi) : (a,b] \to [-\infty, \infty)$ and $M(\varphi) : (a,b] \to (-\infty, \infty]$ will be the functions defined by

$$m(\varphi)(t) = \inf\{\varphi(x) : a<x<t\}$$

and

$$M(\varphi)(t) = \sup\{\varphi(x) : a<x<t\}.$$

If $f : [a,b] \to \mathbf{R}$ then $\bar{f} : (a,b] \to \mathbf{R}$ will be the function defined by $\bar{f}(x) = (f(x) - f(a))/(x-a)$ and $D_+f(a)$ and $D^+f(a)$ will denote the right-lower and right-upper

Dini Derivates of f at a defined by

$$D_{+}f(a)=\sup\ \{m(\tilde{f})(t) : a<t<b\}$$

and

$$D^{+}f(a)=\inf\ \{M(\tilde{f})(t) : a<t<b\}.$$

The left derivates $D_{-}f(b)$ and $D^{-}f(b)$ at b are defined analogously.

It is clear that $m(\varphi)$ is a decreasing function, $M(\varphi)$ is an increasing function and that $m(\varphi)\leq M(\varphi)$.

Hence we have:

$$D_{+}f(a)=\lim_{t\downarrow a} m(\tilde{f})(t),\ D^{+}f(a)=\lim_{t\downarrow a} M(\tilde{f})(t)$$

and

$$D_{+}f(a)\leq D^{+}f(a).$$

The derivation rules to be obtained in this note will usually be stated only for the right-derivates since it should then be obvious that analogous results hold for the left-derivates.

REMARK 1.2. Clearly, for $\alpha\geq 0$, $m(\alpha\varphi)=\alpha m(\varphi)$ and $M(\alpha\varphi)=\alpha M(\varphi)$ while for $\alpha\leq 0$, $m(\alpha\varphi)=\alpha M(\varphi)$ and $M(\alpha\varphi)=\alpha m(\varphi)$. (It is understood that $0\cdot\infty=0\cdot(-\infty)=0$). These observations immediately yield: if $\alpha\geq 0$ then $D(\alpha f)(a)=\alpha Df(a)$ where D denotes D_{+} or D^{+} while if $\alpha\leq 0$ then $D_{+}(\alpha f)(a)=\alpha D^{+}f(a)$ and $D^{+}(\alpha f)(a)=\alpha D_{+}f(a)$. These are, however, particular cases of the more general rules of derivation contained in

THEOREM 2.3. *Before taking up these rules it is necessary to establish the following lemmas regarding the functionals m and M.*

LEMMA 1.3. *If* $\varphi : (a,b]\to\mathbf{R}$ *and* $\phi : (a,b]\to\mathbf{R}$ *then*

$$\text{(i)}\qquad m(\varphi+\phi)\geq m(\varphi)+m(\phi)$$

and

$$\text{(ii)}\qquad M(\varphi+\phi)\leq M(\varphi)+M(\phi).$$

If either φ *is bounded below or* ϕ *is bounded above then there also holds*

$$\text{(iii)}\qquad m(\varphi+\varphi)\leq m(\varphi)+M(\phi)\leq M(\varphi+\phi).$$

PROOF. (i) and (ii) are clear. Using these and the Remark 1.2 above, we have:

$$m(\varphi)=m(\varphi+\phi-\phi)\geq m(\varphi+\phi)+m(-\phi)=m(\varphi+\phi)-M(\phi)$$

and

$$M(\phi)=M(\varphi+\phi-\varphi)\leq M(\varphi+\phi)+M(-\varphi)=M(\varphi+\phi)-m(\varphi)$$

LEMMA 1.4. *Let $f:[a,b]\to\mathbf{R}$, $g:[a,b]\to[0,\infty)$ and suppose that g is continuous. Then*

$$\lim_{t\downarrow a} m(\bar{f}g)(t)=g(a)D_+f(a) \text{ and } \lim_{t\downarrow a} M(\bar{f}g)(t)=g(a)D^+f(a).$$

PROOF. For each $t\in[a,b]$ let

$$\lambda(t)=\min\ \{g(x) : a\leq x\leq t\}$$

and

$$\mu(t)=\max\{g(x) : a\leq x\leq t\}.$$

Then $0\leq\lambda(t)\leq g(x)\leq\mu(t)$ for each $x\in[a,t]$ and $t\in[a,b]$ and $\lambda(t)\uparrow g(a)$ and $\mu(t)\downarrow g(a)$ as $t\downarrow a$. If $\bar{f}(x)\geq 0$ for each $x\in(a,t)$, then

$$\lambda(t)\bar{f}(x)\leq\bar{f}(x)g(x)\leq\mu(t)\bar{f}(x) \tag{1}$$

for all $x\in(a,t)$ and hence taking infimum over all $x\in(a,t)$ we have

$$\lambda(t)m(\bar{f})(t)\leq m(\bar{f}g)(t)\leq\mu(t)m(\bar{f})(t) \tag{2}$$

which on taking limits as $t\downarrow a$ gives:

$$\lim_{t\downarrow a} m(\bar{f}g)(t)=g(a)D_+f(a). \tag{3}$$

If, on the other hand, $\bar{f}(x)<0$ for some $x\in(a,t)$ let $S_t=\{x\in(a,t) : \bar{f}(x)<0\}$. Since $g(x)\geq 0$ for all x, $\inf\{\bar{f}(x)g(x) : x\in(a,t)\}=\inf\{\bar{f}(x)g(x) : x\in S_t\}$ and so too for $\inf\{\lambda(t)\bar{f}(x)\}$ and $\inf\{\mu(t)\bar{f}(x)\}$. For each $x\in S_t$, (1) will be replaced by (1′) in which the inequalities of (1) are reversed. Hence taking infimum over all $x\in(a,t)$, (1′) yields (2′) in which the inequalities of (2) are reversed; taking limits in (2′) as $t\downarrow a$ we again get (3).

This proves the first half of the lemma. The second half is proved similarly (or follows from the first half by replacing f by $-f$).

2. Rules of Derivation

THEOREM 2.1. *Let $f:[a,b]\to\mathbf{R}$, $g:[a,b]\to\mathbf{R}$. Then*

(i) $D_+f(a)+D_+g(a)\leq D_+(f+g)(a)$

(ii) $D^+f(a)+D^+g(a)\geq D^+(f+g)(a)$

If either $D_+f(a)$ or $D^+g(a)$ is finite then there also holds

(iii) $D_+(f+g)(a)\leq D_+f(a)+D^+g(a)\leq D^+(f+g)(a)$.

PROOF. Applying lemma 1.3 to the functions $\bar{f}$ and $\bar{g}$, (i) and (ii) follow immediately. If $D_+f(a)$ is finite then there exists a number $c\in(a,b)$ such that $m(\bar{f})$ is bounded both above and below on $(a,c]$. But then $\bar{f}$ is bounded below on

$(a, c]$ and part (iii) of lemma 1.3 applies to $\tilde{f}$ and $\tilde{g}$ restricted to $(a, c]$. Thus for each t in $(a, c]$,

$m(\tilde{f}+\tilde{g})(t) \leq m(\tilde{f})(t)+M(\tilde{g})(t) \leq M(\tilde{f}+\tilde{g})(t)$. On taking limits as $t \downarrow a$, the assertion (iii) above results. The case of a finite $D^+g(a)$ is similar.

COROLLARY 2.2. *Let* $f: [a, b] \to \mathbf{R}$, $g: [a, b] \to \mathbf{R}$ *and suppose that* f *or* g *has a finite right-derivative at* a. *Then* $D(f+g)(a)=Df(a)+Dg(a)$ *where* D *denotes* D_+ *or* D^+.

PROOF. If g has a right derivative $g_+'(a)$ at a, then $D_+g(a)=D^+g(a)=g_+'(a)$ From (i) and (iii) of Theorem 2.1, we have:

$D_+f(a)+g_+'(a) \leq D_+(f+g)(a) \leq D_+f(a)+g_+'(a)$ while from (ii) and (iii) of the same theorem, $D^+f(a)+g_+'(a) \geq D^+(f+g)(a) \geq D^+f(a)+g_+'(a)$.

THEOREM 2.3. *Let* $f: [a, b] \to \mathbf{R}$, $g: [a, b] \to [0, \infty)$ *and suppose that* g *is continuous. If* $f(a) \geq 0$, *then*

(i) $g(a)D_+f(a)+f(a)D_+g(a) \leq D_+(fg)(a)$

(ii) $g(a)D^+f(a)+f(a)D^+g(a) \geq D^+(fg)(a)$:

if in addition one of $D_+f(a)$, $D^+g(a)$ *is finite there holds also*

(iii) $D_+(fg)(a) \leq g(a)D_+f(a)+f(a)D^+g(a) \leq D^+(fg)(a)$

while if one of $D^+f(a)$, $D_+g(a)$ *is finite there holds*

(iv) $D_+(fg)(a) \leq g(a)D^+f(a)+f(a)D_+g(a) \leq D^+(fg)(a)$.

If $f(a) \leq 0$ *the above statements hold with* $f(a)D^+g(a)$ *replaced by* $f(a)D_+g(a)$ *and* $f(a)D_+g(a)$ *replaced by* $f(a)D^+g(a)$.

PROOF. Set $\alpha=f(a)$ and suppose $f(a) \geq 0$. From the identity $\widetilde{fg}=\tilde{f}g+\alpha\tilde{g}$ we have, by 1.2 and 1.3 $m(\widetilde{fg}) \geq m(\tilde{f}g)+m(\alpha\tilde{g})=m(\tilde{f}g)+\alpha m(\tilde{g})$. Evaluating at $t \in (a, b)$ and taking limits as $t \downarrow a$, it follows by lemma 1.4 that

$$D_+(fg)(a) \geq g(a)D_+f(a)+\alpha D_+g(a).$$

This proves (i). Parts (ii), (iii), (iv) are proved similarly using appropriate parts of 1.2, 1.3, 1.4 and resorting to the same kind of arguments as in the proof of Theorem 2.1(iii). The case of $f(a)<0$ follows by applying the results so far established to the function $-f$.

COROLLARY 2.4. *Let* $f: [a, b] \to \mathbf{R}$, $g: [a, b] \to [0, \infty)$ *and suppose that* g *is* *continuous and that f or g has a finite right-derivative at a. If $f(a)\geq 0$ then $D(fg)(a)=f(a)Dg(a)+g(a)Df(a)$ where D denotes D_+ or D^+. If $f(a)\leq 0$ then the same result holds with $f(a)D_+g(a)$ replaced by $f(a)\ D^+g(a)$ and $f(a)D^+g(a)$ replaced by $f(a)D_+g(a)$.*

PROOF. Suppose $f(a)\geq 0$. If g has a finite right-derivative $g_+'(a)$ at a, then $D_+g(a)=D^+g(a)=g_+'(a)$. From (i) and (iii) of theorem 2.3,

$$g(a)D_+f(a)+f(a)D_+g(a)\leq D_+(fg)(a)\leq g(a)D_+f(a)+f(a)D^+g(a)$$
$$=g(a)D_+f(a)+f(a)D_+g(a)$$

This proves the result for $D=D_+$. Similarly for $D=D^+$ using (ii) and (iv) of the theorem. If f has a finite right-derivative at a then the result follows from (i), (iv) and (ii), (iii) of the theorem. The case of $f(a)<0$ follows from what is already proved by applying the results to $-f$.

LEMMA 2.5. *Let g : $[a,b]\rightarrow\mathbf{R}$ be continuous and suppose that $g(x)\neq 0$ for any x. Then*

(i) $D^+\left(\frac{1}{g}\right)(a)=\frac{-1}{(g(a))^2}\ D_+g(a)$

and

(ii) $D_+\left(\frac{1}{g}\right)(a)=\frac{-1}{(g(a))^2}D^+g(a)$.

PROOF. Either $g(x)>0$ for all $x\in[a,b]$ or $g(x)<0$ for all $x\in[a,b]$. Suppose $g(x)>0$ for all x and let $f=\frac{1}{g}$, $\alpha=g(a)$. Now $fg=1$ gives $\tilde{g}f+\alpha\tilde{f}=\tilde{1}=0$ so that $M(\tilde{f})=M\left(-\frac{1}{\alpha}\tilde{g}f\right)=-\frac{1}{\alpha}m(\tilde{g}f)$. Evaluating at $t\in(a,b)$ and taking limits as $t\downarrow a$, it follows from 1.4 that $D^+f(a)=-\frac{1}{\alpha}f(a)D_+g(a)$. This proves (i) ; (ii) is proved similarly. The case of $g(x)<0$ for all x follows by applying the results to $-g$.

This lemma together with theorem 2.3 and corollary 2.4 yield quotient rules for derivates which are stated in the following theorem and its corollary of which the proofs are omitted.

THEOREM 2.6. *Let f: $[a,b]\rightarrow\mathbf{R}$, g : $[a,b]\rightarrow(0,\infty)$ and suppose that g is continuous. If $f(a)\geq 0$ then*

(i) $(g(a)D_+f(a)-f(a)D^+g(a))/(g(a))^2\leq D_+(f/g)(a)$

(ii) $(g(a)D^+f(a)-f(a)D_+g(a))/(g(a))^2\geq D^+(f/g)(a)$;

if in addition one of $D_+f(a)$, $D_+g(a)$ *is finite then there also holds*

(iii) $D_+(f/g)(a)\leq(g(a)D_+f(a)-f(a)D_+g(a))/(g(a))^2\leq D^+(f/g)(a)$

while if one of $D^+f(a), D^+g(a)$ *is finite then there also holds*

(iv) $D_+(f/g)(a)\leq(g(a)D^+f(a)-f(a)D^+g(a))/(g(a))^2\leq D^+(f/g)(a)$.

If $f(a)\leq 0$ *the same results as above hold with* $f(a)\ D^+g(a)$ *replaced by* $f(a)D_+g(a)$ *and* $f(a)D_+g(a)$ *replaced by* $f(a)D^+g(a)$.

COROLLARY 2.7. *Let* $f: [a, b]\to\mathbf{R}$, $g: [a,b]\to(0,\infty)$ *and suppose that* g *is continuous and that* f *or* g *has a finite right-derivative at* a. *If* $f(a)\leq 0$ *then* $D(f/g)(a)=(g(a)Df(a)-f(a)Dg(a))/(g(a))^2$ *where* D *denotes* D_+ *or* D^+. *If* $f(a)\geq 0$ *then the same result holds with* $f(a)D^+g(a)$ *replaced by* $f(a)D_+g(a)$ *and* $f(a)D_+g(a)$ *replaced by* $f(a)D^+g(a)$.

3. Zygmund's lemmas

Throughout this section f is a continuous real-valued function defined on an interval J of the real line and I denotes the interior of this interval.

LEMMA 3.1. *If* $\alpha\in\mathbf{R}$ *is such that* $D_+f(x)\leq\alpha$ *for all* $x\in I$ *(resp.* $D_-f(x)\leq\alpha$ *for all* $x\in I$*) then for all* a,b *in* J *with* $a\leq b$, *there holds* $f(b)-f(a)\leq\alpha(b-a)$.

LEMMA 3.2. *If* $\alpha\in\mathbf{R}$ *is such that* $D^+f(x)\geq\alpha$ *for all* $x\in I$ *(resp.* $D^-f(x)\geq\alpha$ *for all* $x\in I$*) then for all* a,b *in* J *with* $a\leq b$, *there holds* $f(b)-f(a)\geq\alpha(b-a)$.

COROLLARY 3.3. *If* f *is increasing on* J *then* $D_+f(x)\geq 0$ *and* $D_-f(x)\geq 0$ *for each* $x\in I$ *; conversely if* $D^+f(x)\geq 0$ *for each* $x\in I$ *or* $D^-f(x)\geq 0$ *for each* $x\in I$, *then* f *is increasing on* J.

COROLLARY 3.4. *If* f *is decreasing on* J *then* $D^+f(x)\leq 0$ *and* $D^-f(x)\leq 0$ *for each* $x\in I$ *: conversely if* $D_+f(x)\leq 0$ *for each* $x\in I$ *or* $D_-f(x)\leq 0$ *for each* $x\in I$, *then* f *is decreasing on* J.

COROLLARY 3.5. *Let* α,β *be elements of* $[-\infty,\infty]$. *If one of the four Dini Derivates* Df *of* f *satisfies* $\alpha\leq Df(x)\leq\beta$ *for all* $x\in I$, *then all the four do.*

4. Applications

A. Nagumo's Uniqueness theorem.

A complete statement of this result is as follows.

THEOREM. *Let a, b be positive numbers, $t_0 \in \mathbf{R}$, $x_0 \in \mathbf{R}^n$ and $D=\{(t, x) \in \mathbf{R} \times \mathbf{R}^n :$ $|t-t_0| \leq a,\ \|x-x_0\| \leq b\}$. If $f: D \rightarrow \mathbf{R}^n$ satisfies $\|f(t,x)-f(t,y)\|\,|t-t_0| \leq \|x-y\|$ for all (t,x) (t,y), in D then the initial value problem*

$$x'(t)=f(t, x(t)),\quad x(t_0)=x_0$$

has at most one solution on any interval containing t_0.

The usual method of proof [1] is to replace the initial value problem by an integral equation and appeal to a functional inequality resembling some form of a Gronwall's lemma. Although this is justied even when f is not assumed to be continuous, the following is an alternative favoured in [5].

PROOF. Let x, y be solutions on $[t_0-\alpha,\ t_0+\beta]$ where $0 \leq \alpha \leq a$, $0 \leq \beta \leq a$ and define $g:[0, \beta] \rightarrow [0, \infty)$ by $g(t)=\|x(t_0+t)-y(t_0+t)\|$. Then for s, t in $[0, \beta]$ with $s \neq t$,

$$\left|\frac{g(s)-g(t)}{s-t}\right| \leq \left\|\frac{x(t_0+s)-x(t_0+t)}{s-t}-\frac{y(t_0+s)-y(t_0+t)}{s-t}\right\|$$

so that for each $t \in (0, \beta)$, $|Dg(t)| \leq \|x'(t_0+t)-y'(t_0+t)\|$ where Dg denotes any of the Dini Derivates of g. But

$$\|x'(t_0+t)-y'(t_0+t)\|=\|f(t_0+t,\ x(t_0+t))-f(t_0+t,\ y(t_0+t))\|$$
$$\leq \frac{1}{t}\|x(t_0+t)-y(t_0+t)\|=g(t)/t$$

Hence we have $Dg(t) \leq g(t)/t$ which gives

$$D(g(t)/t)=\frac{1}{t} Dg(t)-\frac{1}{t^2} g(t)=\frac{1}{t}(Dg(t)-g(t)/t) \leq 0.$$

This shows that $g(t)/t$ is a decreasing function and hence $g(t)/t \leq \lim_{t \downarrow 0} g(t)/t$ $=D^+ g(0)$. But again,

$$|D^+ g(0)| \leq \|x'(t_0)-y'(t_0)\|=\|f(t_0, x(t_0))-f(t_0, y(t_0))\|$$
$$=\|f(t_0, x_0)-f(t_0, x_0)\|=0.$$

Thus $g(t)/t \leq 0$ which gives $g(t)=0$ for all $t \in (0, \beta)$. Consequently, $x=y$ on $(t_0, t_0+\beta)$ and by continuity this equality holds on $[t_0, t_0+\beta]$; a similar argument applies to the interval $[t_0-\alpha, t_0]$.

B. Lyapunov's second instability theorem.

Consider again the differential system $x'(t)=f(t, x(t))$ where $f:[0, \infty) \times B(r)$ $\rightarrow \mathbf{R}^n$ is continuous with $f(t, 0)=0$ for all $t \geq 0$, $B(r)$ being the open ball of radius r around the origin in $\mathbf{R}^n$. For $W:[0, \infty) \times B(\varepsilon) \rightarrow \mathbf{R}$ where $0<\varepsilon \leq r$, define

$\dot{W}: [0,\infty)\times B(\varepsilon)\to[-\infty,\infty]$ by

$$\dot{W}(t,x)=\overline{\lim_{h\downarrow 0}}\ \frac{1}{h}(W(t+h,x+hf(t,x))-W(t,x))\ ;$$

call the function $\dot{W}$ locally Lipschitzian iff for each $(t,x)\in[0,\infty)\times B(\varepsilon)$ there exists a number $K>0$ and a neighbourhood N of (t,x) such that $|W(s,y)-W(s,z)|\leq K\|y-z\|$ for all (s,y), (s,z) in N. It can be shown that if W is locally Lipschitzian and $x:[0,\infty)\to\mathbf{R}^n$ is a solution of the differential system with $|x(t)|<\varepsilon$ for all $t\geq 0$ then $\dot{W}(t,x(t))=D^+V(t)$ where $V(t)=W(t,x(t))$.

One version of the second instability theorem may now be stated thus.

THEOREM 4.1. *If there exists a bounded locally Lipschitzian* $W: [0,\infty)\times B(\varepsilon)\to\mathbf{R}$, $(0<\varepsilon\leq r)$ *and a continuous* $\lambda: [0,\infty)\to[0,\infty)$ *with* $\int_0^\infty \lambda(t)dt=\infty$ *such that*

(i) $\dot{W}(t,x)\geq\lambda(t)W(t,x)$ *for all* (t,x) *in* $[0,\infty)\times B(\varepsilon)$

(ii) *for each* $\delta>0$ *there exists* $x_0\in\mathbf{R}^n$ *with* $\|x_0\|<\delta$ *and* $W(0,x_0)>0$, *then the "zero-solution" is unstable.*

NOTE. In the usual version [3] the local Lipschitzian hypothesis is replaced by the much stronger requirement of Frechet differentiability.

However, the following proof does not require W to be Frechet differentiable)

PROOF. Suppose that the zero solution is stable. Choose $\delta\in(0,r)$ such that each solution x with $\|x(0)\|<\delta$ is defined for all $t\geq 0$ and satisfies $\|x(t)\|<\varepsilon$ for all $t\geq 0$. Now choose a point $x_0\in\mathbf{R}^n$ such that (ii) holds. Let x be the solution of the initial value problem $x'(t)=f(t,x(t))$, $x(0)=x_0$. Now x is defined for all $t\geq 0$ and $\|x(t)\|<\varepsilon$ for all $t\geq 0$. Writing $\mu(t)=\exp\left(-\int_0^t\lambda(s)ds\right)$ we have,

$$D^+(\mu(t)W(t,x(t)))=\mu(t)D^+W(t,x(t))-\lambda(t)\mu(t)W(t,x(t))$$
$$=\mu(t)(\dot{W}(t,x(t))-\lambda(t)W(t,x(t)))\geq 0 \text{ by (i).}$$

Therefore, $\mu(t)W(t,x(t))$ is an increasing function of t and hence for all $t\geq 0$,

$$\mu(t)\dot{W}(t,x(t))\geq\mu(0)W(0,x(0))=W(0,x_0)$$

so that $W(t,x(t))\geq W(0,x_0)\exp\left(\int_0^t\lambda(s)ds\right)$. Since

$$W(0,x_0)>0 \text{ and } \int_0^\infty\lambda(s)ds=\infty,$$

$W(0,x_0)\exp\left(\int_0^t\lambda(s)ds\right)\to\infty$ as $t\to\infty$. Hence it follows that $W(t,x(t))\to\infty$ as

$t \to \infty$ and this contradicts the boundedness of W. The proof is complete.

C. Subadditive functions.

One of the results in [2] regarding the differentiability and growth properties of a subadditive function $f: \mathbf{R} \to \mathbf{R}$ states that if $\sup\{f(t)/t : t>0\}=B$ and $\inf\{f(t)/t : t<0\}=A$ then $D^+f(t) \leq B$, $D^-f(t) \leq B$, $D_+f(t) \geq A$ and $D_-f(t) \geq A$ for all t; if A and B are finite then f is absolutely continuous and if further A and B are equal then $f(t)=At$ for all t. Nothing has been stated in this context regarding the growth properties of f in case A and B are finite but unequal (and hence necessarily $-\infty<A<B<\infty$). However, the following result readily holds.

THEOREM 4.2. *If* $f: \mathbf{R} \to \mathbf{R}$ *is subadditive and if* $A=\inf\{f(t)/t : t<0\}$ *and* $B=\sup\{f(t)/t : t>0\}$ *are finite then*

$$At \leq f(t) \leq Bt \text{ for } t \geq 0 \text{ and } Bt \leq f(t) \leq At \text{ for } t \leq 0.$$

PROOF. From the results mentioned above, f is continuous and hence $D^+(f(t)-tB)=D^+f(t)-B \leq 0$ for $t \geq 0$ so that $f(t)-tB$ is decreasing on $[0,\infty)$. Hence $f(t)-tB \leq f(0)$ for $t \geq 0$. This gives $f(t) \leq tB$ for $t \geq 0$ since $f(0)=0$ by theorem 7.11.1 of [2]. The remaining ineqalities are proved similarly.

Another result of [2] states that if $f: (0,\infty) \to \mathbf{R}$ is such that $f(t)/t$ is a decreasing function of t then f is subadditive. Using this Rathore [4] shows that if $f: (a,\infty) \to \mathbf{R}$ is differentiable where $a \geq 0$ and if $xf'(x)<f(x)$ for all x in (a,∞) then f is subadditive. However, these hypotheses on f imposed in [4] are too restrictive; it is just as easy to prove the following.

THEOREM 4.3. *Let* $a \geq 0$ *and* $f: (a,\infty) \to \mathbf{R}$. *If* $xD_+f(x) \leq f(x)$ *for all* x *in* (a,∞) *or* $xD_-f(x) \leq f(x)$ *for all* x *in* (a,∞) *then* f *is subadditive.*

PROOF. Suppose $xD_+f(x) \leq f(x)$ for all x in (a,∞). Then $D_+(f(x)/x) = \frac{1}{x^2}(xD_+f(x)-f(x)) \leq 0$ and hence $f(x)/x$ is decreasing; similarly for $xD_-f(x) \leq f(x)$.

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