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SEMI-SYMMETRIC METRIC CONNECTION IN AN ALMOST CONTACT METRIC MANIFOLD

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Friedmann and Schouten [1] introduced the idea of semi-symmetric linear connection in a differentiable manifold. Hayden [2] introduced the idea of metric connection with torsion tensor in a Riemannian manifold. Recently, Yano [3] and Imai [4] studied the properties of semi-symmetric metric connection in a Riemannian manifold.

The purpose of this paper is to introduce the idea of semi-symmetric metric connection in an almost contact metric manifold and to study its properties.

1. Preliminaries

Let there exist in an $n(=2m+1)$ dimensional real differentiable manifold of differentiability class C^∞ a C^∞ vector-valued linear function F, a C^∞ vector field T and a C^∞ 1-form A satisfying

(1.1) a) $\overline{X} \overset{\text{def}}{=\!=} F(X)$, b) $A(T)=L$, c) $A(\overline{X})=0$, d) $\overline{T}=0$, e) $\overline{\overline{X}}+X=A(X)T$,

for an arbitrary vector field X. Then M_n is called an almost contact manifold and the structure (F,T,A) is called an almost contact structure.

Recently, Mishra [5] has proved that (1.1) e) alone defines an almost contact structure in a real differentiable manifold of differentiability class C^∞.

Let the almost contact metric manifold M_n be endowed with the non-singular metric tensor g satisfying

(1.2) $$g(\overline{X},\overline{Y})=g(X,Y)-A(X)A(Y).$$

Then the manifold is called an almost contact metric manifold. Putting T for X in (1.2) and using (1.1) we obtain

(1.3) $$g(Y,T)=A(Y).$$

In an almost contact metric manifold the Nijenhuis tensor is given by

(1.4) a) $N(X,Y)=(D_{\overline{X}}F)(Y)-(D_{\overline{Y}}F)(X)-\overline{(D_XF)(Y)}+\overline{(D_YF)(X)}$,

whence

b) $'N(X,Y,Z)=(D_{\overline{X}}{}'F)(Y,Z)-(D_{\overline{Y}}{}'F)(X,Z)+(D_X{}'F)(Y,\overline{Z})-(D_Y{}'F)(X,\overline{Z})$,

c) $$'N(\bar{Z},\bar{Y},\bar{X})=(D_{\bar{\bar{Z}}}'F)(\bar{Y},\bar{X})+(D_{\bar{Y}}'F)(\bar{X},\bar{\bar{Z}})+(D_{\bar{X}}'F)(\bar{\bar{\bar{Z}}},Y)-(D_{\bar{\bar{Y}}}'F)(\bar{Z},\bar{\bar{X}})$$
$$-(D_{\bar{\bar{Z}}}'F)(\bar{\bar{X}},\bar{Y})-(D_{\bar{X}}'F)(\bar{Y},\bar{\bar{Z}})+2(D_{\bar{X}}'F)(\bar{\bar{Y}},\bar{Z}).$$

In consequence of (1.1) c), (1.4) a) gives

(1.5) a) $$A(N(X,Y))=A([\bar{X},\bar{Y}]),$$

b) $$A(N(\bar{X},\bar{Y}))=A([\bar{\bar{X}},\bar{\bar{Y}}]).$$

If in an almost contact metric manifold M_n

(1.6) $$'F(X,Y)=(dA)(X,Y),$$

the almost contact metric manifold is called an almost Sasakianmanifold (1.6)a) is equivalent to

(1.7) $$(D_X'F)(Y,Z)-(D_Y'F)(X,Z)+(D_Z'F)(X,Y)=0,$$

where D is a Riemannian connection.

2. Semi-symmetric metric connection.

Let D be a Riemannian connection in an almost contact metric manifold and B another affine connection satisfying

(2.1) $$(B_Xg)(Y,Z)=0.$$

The torsion tensor of B is given by

$$S(X,Y)=B_XY-B_YX-[X,Y].$$

DEFINITION 2.1. If the torsion tensor S satisfies

(2.2) $$S(X,Y)=A(Y)X-A(X)Y,$$

the connection B will be called *semi-symmetric metric connection.*

Let us put

(2.3) $$B_XY=D_XY+H(X,Y).$$

Consequently

(2.4) $$S(X,Y)=H(X,Y)-H(Y,X).$$

Let us put

(2.5) a) $'S(X,Y,Z)=g(S(X,Y),Z)$, b) $'H(X,Y,Z)=g(H(X,Y),Z)$.

Then

(2.6) $$'S(X,Y,Z)='H(X,Y,Z)-'H(Y,X,Z).$$

LEMMA 2.1. *Let D be a Riemannian connection in M_n and B a semi-symmetric metric connection satisfying* (2.1). *Then*

(2.7) $$'H(Y,Z,X)='S(X,Z,Y).$$

PROOF. From (2.1), (2.3) and (2.5) b) we have

(2.8) $$'H(X,Y,Z)+'H(X,Z,Y)=0.$$

In view of (2.6) and (2.8) we obtain

(2.9) $$'S(X,Y,Z)='H(X,Y,Z)+'H(Y,Z,X).$$

From (2.9) we obtain

(2.10) $$2'H(Y,Z,X)='S(X,Y,Z)+'S(Y,Z,X)-'S(Z,X,Y).$$

In view of (2.2) and (2.5) a) the above equation reduces to (2.7). This completes the proof.

In view of (2.2), (2.5)a) and lemma 2.1, the equation (2.3) becomes

(2.11) $$B_X Y=D_X Y+A(Y)+A(Y)X-g(X,Y)T.$$

Equation (2.11) also defines semi-symmetric metric connection in an almost contact metric manifold.

For the covariant differentiation of 1-form A we have

(2.12) $$(B_X A)(Y)=(D_X A)(Y)-A(X)A(Y)+g(X,Y).$$

Such a linear connection B will be called semi-symmetric metric connection.

THEOREM 2.1. *Let D be a Riemannian connection in M_n and B be a semi-symmetric metric connection satisfying*

(2.13) a) $$(B_X 'F)(Y,Z)=0,$$ b) $$A(B_{\bar{X}}\bar{\bar{Y}}-B_{\bar{Y}}\bar{\bar{X}})=A(B_{\bar{X}}\bar{Y}-B_{\bar{Y}}\bar{X}).$$

Then the almost contact metric manifold is completely integrable.

PROOF. We know [6] that when the almost contact manifold is completely integrable

(2.14) a) $$\overline{N(\bar{X},\bar{Y})}=0,$$ b) $$A(N(X,Y))=A(N(\bar{X},\bar{Y}))$$

For an almost contact metric manifold (2.14)a) is equivalent to

(2.15) $$'N(\bar{X},\bar{Y},\bar{Z})=0.$$

In consequence of (2.13)a) we have

(2.16) $$(D_X 'F)(\bar{Y},\bar{Z})='H(X,\bar{Z},\bar{\bar{Y}})-'H(X,\bar{Y},\bar{\bar{Z}}).$$

Barring X and Z in (2.16) and using (1.1) we obtain

(2.17) $$(D_{\bar{X}} 'F)(\bar{Y},\bar{\bar{Z}})='H(\bar{X},\bar{\bar{Z}},\bar{\bar{Y}})+'H(\bar{X},\bar{Y},\bar{Z}).$$

Also, from (2.2), (2.5)a) and lemma 2.1, we obtain

(2.18) $$'H(\bar{X},\bar{Y},\bar{Z})=0.$$

Using (2.18) in (2.17) we obtain

(2.19) $$(D_{\bar{X}} 'F)(\bar{Y},\bar{\bar{Z}})=0.$$

Using (2.19) in (1.4) c) we obtain (2.15). Equation (2.14) follows immediately from (2.13)b), in consequence of (1.5) a), b).

Thus the proof is complete.

COROLLARY 2.1. *When the affine connection satisfies* (2.13) a) :

(2.20) $$'S(Z,Y,\overline{X})=(D_X{}'F)(Y,Z).$$

Consequently, when M_n is an almost Sasakian manifold :

(2.21) $$S(X,Z,\overline{Y})+S(Y,X,\overline{Z})+S(Z,Y,\overline{X})=0.$$

PROOF. In consequence of (2.13)a) we have

$$'F(B_XY,Z)+'F(Y,B_XZ)=(D_X{}'F)(Y,Z)+'F(D_XY,Z)+'F(Y,D_XZ).$$

Using (2.3) in this equation, we get

(2.22) $$(D_X{}'F)(Y,Z)='H(X,Z,\overline{Y})-'H(X,Y,\overline{Z})$$

From (2.2), (2.5)a), (2.22) and lemma 2.1, we obtain (2,20), (2.22) follows immediately from (1.7) and (2.20).

THEOREM 2.2. *Let D be a Riemannian connection in an almost contact metric manifold and B be a semi-symmetric connexion satisfying* $(B_XA)(Y)-(B_YA)(X)='F(X,Y)$. *Then M_n is an almost Sasakian manifold.*

The proof is obvious.

3. Curvature tensor of a semi-symmetric metric connection

Let R be the curvature tensor with respect to the connection B:

(3.1) $$R(X,Y,Z)=B_XB_YZ-B_YB_XZ-B_{[X,Y]}Z,$$

and K be the curvature tensor with respect to the connection D.

(3.2) $$K(X,Y,Z)=D_XD_YZ-D_YD_XZ-D_{[X,Y]}Z.$$

A manifold satisfying

(3.3) $$R(X,Y,Z)=0,$$

and

(3.4) $$(B_XS)(Y,Z)=0,$$

is called a group manifold [3]. Equation (3.4) implies

(3.5) $$(D_XA)(Y)-A(X)A(Y)+g(X,Y)=0,$$

where we have used (2.2) and (2.12).

THEOREM 3.1. *If the almost contact metric admits a semi-symmetric metric connection for which the manifold is a group manifold, then the almost contact metric is of constant curvature.*

PROOF. In view of (1.1), (2.11), (3.1) and (3.2) we have after some calculations

(3.6) $$'R(X,Y,Z,W)='K(X,Y,Z,W)-(g(Y,Z)g(X,W)-g(X,Z)g(Y,W))$$
$$-g(X,W)((D_YA)(Z)-A(Y)A(Z))$$
$$+g(Y,W)((D_XA)(Z)-A(X)A(Z))$$
$$-g(Y,Z)((D_XA)(W)-A(X)A(W))$$
$$+g(X,Z)((D_YA)(W)-A(Y)A(W)),$$

where $'R(X,Y,Z,W)=g(R(X,Y,Z),W)$ and $'K(X,Y,Z,W)=g(K(X,Y,Z),W)$. In view of (3.4), (3.5) and (3.6) we have

(3.7) $$'K(X,Y,Z,W)=g(X,Z)g(Y,W)-g(Y,Z)g(X,W).$$

This completes the proof.

THEOREM 3.2. *An almost contact metric manifold with semi-symmetric metric connection whose curvature tensor vanishes is of constant curvature* 1 *iff* $(D_XA)(Y)=A(X)A(Y)$, *where X and Y are arbitrary vector fields.*

PROOF. Putting $'R(X,Y,Z,W)=0$ in (3.6), we get

(3.8) $$'K(X,Y,Z,W)=g(Y,Z)g(X,W)-g(X,Z)g(Y,W)$$

iff

$$(D_XA)(Y)=A(X)A(Y).$$

4. The induced connection

Let M_{2m-1} be submanifold of M_{2m+1} and let $c: M_{2m-1} \longrightarrow M_{2m+1}$ be the inclusion map such that

$$d \in M_{2m-1} \longrightarrow cd \in M_{2m+1}.$$

c induces a linear transformation (Jacobian map) J

$$J: T'_{(2m-1)} \longrightarrow T'_{(2m+1)},$$

where $T'_{(2m-1)}$ is the tangent space to M_{2m-1} at a point d and $T'_{(2m+1)}$ is the tangent space to M_{2m+1} at cd, such that

$$\tilde{X} \text{ in } M_{2m-1} \text{ at } d \longrightarrow J\tilde{X} \text{ in } M_{2m+1} \text{ at } cd.$$

Let $\tilde{g}$ be the induced metric tensor in M_{2m-1}. Then we have

(4.1) $$\tilde{g}(\tilde{X},\tilde{Y})=(g(J\tilde{X},J\tilde{Y}))b$$

We now suppose that the almost contact metric manifold M_{2m+1} admits a semi-symmetric metric connection given by

(4.2) $$B_XY=D_XY+A(X)Y-g(X,Y)T,$$

where X and Y are arbitrary vector field of M_{2m+1}. Let us put

(4.3) $$T=Jt+\rho M+\sigma N,$$

where t is a C^∞ vector field in M_{2m-1} and M and N are unit normal vectors to M_{2m-1}.

Denoting by $\dot{D}$ the connection induced on the sub-manifold from D, we have the Gauss equation

(4.4) $$D_{JX}J\tilde{Y}=J(\dot{D}_X\tilde{Y})+h(\tilde{X},\tilde{Y})M+k(\tilde{X},\tilde{Y})N,$$

where h and k are symmetric bilinear functions in M_{2m-1}. Similarly we have

(4.5) $$B_{J\dot{X}}J\tilde{Y}=J(\dot{B}_{\tilde{X}}\tilde{Y})+m(\tilde{X},\tilde{Y})M+n(\tilde{X},\tilde{Y})N,$$

where B is the connection induced on the submanifold from B and m and n are symmetric bilinear functions in M_{2m-1}. From (4.2) we have

$$B_{JX}J\tilde{Y}=D_{J\dot{X}}J\tilde{Y}+A(J\tilde{Y})B\tilde{X}-g(J\tilde{X},J\tilde{Y})T,$$

and hence, using (4.4) and (4.5), we find

(4.6) $$\begin{aligned}&J(\dot{B}_X\tilde{Y})+m(\tilde{X},\tilde{Y})M+n(\tilde{X},\tilde{Y})N\\&=J(\dot{D}_X\tilde{Y})+h(\tilde{X},\tilde{Y})M+k(\tilde{X},\tilde{Y})N\\&+a(\tilde{Y})J\tilde{X}-\tilde{g}(\tilde{X},\tilde{Y})\ (Jt+PM+N),\end{aligned}$$

where $\tilde{g}(\tilde{Y},t)\overset{\text{def}}{=\!=}a(Y)$. This gives

$$\dot{B}_{\tilde{X}}\tilde{Y}=\dot{D}_{\tilde{X}}\tilde{Y}+a(\tilde{Y})\tilde{X}-\tilde{g}(\tilde{X},\tilde{Y})t$$

iff

(4.7) a) $$m(\tilde{X},\tilde{Y})=h(\tilde{X},\tilde{Y})-\rho\tilde{g}(\tilde{X},\tilde{Y}),$$

b) $$n(\tilde{X},\tilde{Y})=k(\tilde{X},\tilde{Y})-\sigma g(\tilde{X},\tilde{Y}).$$

Thus we have

THEOREM 4.1. *The connection induced on a sub-manifold of an almost contact metric manifold with a semi-symmetric metric connection with respect to the unit normal vectors M and N is also a semi-symmetric one iff* (4.7) a), b) *hold.*

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