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ON THE MEANS OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES OF SMALL ORDER

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1. Introduction

Several relations involving the various mean values of entire functions of several complex variables of finite order have been discussed in [1,2,5,6,7,8]. The aim of this paper is to concentrate on the results concerning these means along with two new means $m^*_{\delta,\lambda,\mu}(r_1,r_2)$ and $g^*_{\lambda,\mu}(r_1,r_2)$ [for definition see §2], introduced in this paper, for a class of functions of very small order, particularly of order zero. In section 2, apart from the notations and terminology, we introduce the concepts of the logarithmic order and the lower logarithmic order for this class of functions. In section 3, we state the results of this paper whereas the remaining sections have been devoted to the proofs of our results.

2. Notations and terminology

Let[1)]

$$f(z_1, z_2)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn}\, z_1^m\, z_2^n,$$

be an entire function of two complex variables z_1 and z_2. Let the maximum modulus of the function $f(z_1,z_2)$ be defined as:

$$M(r_1,r_2)=\max\{|f(z_1,z_2)| : |z_i|\le r_i, i=1,2\}.$$

The finite order ρ, of $f(z_1,z_2)$ is defined as (see [3], 219)

$$\limsup_{r_1,r_2\to\infty}\frac{\log\log M(r_1,r_2)}{\log(r_1r_2)}=\rho.$$

Throughout this paper, we consider a class of functions of very small order, particularly of order zero. To have a more precise description of the growth relations for such a class, we define, on the lines of Dzrbasyan [4], the loga-

1) This research work is supported partially by the University Grants Commission (India) For simplicity we consider only two variables, though the results can easily be extended to several variables.

rithmic orders ρ^*_i of $f(z_1, z_2)$ with respect to the variables $z_i (i=1,2)$ as:

$$\limsup_{r_j\to\infty} \limsup_{r_i\to\infty} \left\{\frac{\log\log^+ M(r_1,r_2)}{\log\log r_i}\right\}=\rho^*_i, \quad (i\neq j : i,j=1,2)$$

where $\log^+ x=\max(\log x, 1)$.

Further, let $\rho^*=\max(\rho^*_1, \rho^*_2)$. Then

$$\limsup_{r_1,r_2\to\infty} \frac{\log\log^+ M(r_1,r_2)}{\log\log(r_1r_2)}=\rho^*,$$

and $f(z_1, z_2)$ is said to be of finite L-order (logarithmic order) ρ^*.

Similarly, we can define the lower L-order, λ^*, of $f(z_1, z_2)$ as

$$\liminf_{r_1,r_2\to\infty} \frac{\log\log^+ M(r_1,r_2)}{\log\log(r_1r_2)}=\lambda^*,$$

where $\lambda^*=\min(\lambda^*_1, \lambda^*_2)$, and

$$\lambda^*_i=\liminf_{r_j\to\infty} \liminf_{r_i\to\infty} \left\{\frac{\log\log^+ M(r_1,r_2)}{\log\log r_i}\right\}, \quad (i\neq j : i,j=1,2).$$

Let us consider the following means of $f(z_1, z_2)$:

$$(2.1)\qquad I_\delta(r_1,r_2)=\left(\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} |f(r_1e^{i\theta_1}, r_2e^{i\theta_2})|^\delta d\theta_1 d\theta_2\right)^{\frac{1}{\delta}},$$

$$(2.2)\qquad m_{\delta,\lambda,\mu}(r_1,r_2)=\frac{1}{r_1^{\lambda+1}r_2^{\mu+1}}\int_0^{r_1}\int_0^{r_2} x_1^{\lambda}x_2^{\mu} I_\delta(x_1, x_2)dx_1dx_2,$$

$$(2.3)\qquad m^*_{\delta,\lambda,\mu}(r_1,r_2)=\frac{1}{(\log r_1)^{\lambda+1}(\log r_2)^{\mu+1}}\times$$

$$\int_1^{r_1}\int_1^{r_2} (\log x_1)^{\lambda}(\log x_2)^{\mu} I_\delta(x_1,x_2)\ \frac{dx_1}{x_1}\ \frac{dx_2}{x_2},$$

$$(2.4)\qquad G(r_1,r_2)=\exp\left\{\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \log^+ |f(r_1e^{i\theta_1},\ r_2e^{i\theta_2})|d\theta_1 d\theta_2\right\},$$

$$(2.5)\qquad g_{\lambda,\mu}(r_1,r_2)=\exp\left\{\frac{(\lambda+1)(\mu+1)}{r_1^{\lambda+1}r_2^{\mu+1}}\int_0^{r_1}\int_0^{r_2} x_1^{\lambda}x_2^{\mu}\log\ G(x_1,x_2)dx_1dx_2\right\}$$

and

(2.6) $$g^*_{\lambda,\mu}(r_1,r_2)=\exp\Big\{\frac{(\lambda+1)(\mu+1)}{(\log r_1)^{\lambda+1}(\log r_2)^{\mu+1}}\times \int_1^{r_1}\int_1^{r_2}(\log x_1)^{\lambda}(\log x_2)^{\mu}\log G(x_1,x_2)\,\frac{dx_1}{x_1}\,\frac{dx_2}{x_2}\Big\},$$

where $\delta\geq 1$ and $-1<\lambda,\ \mu<\infty$.

3. Statement of the results

THEOREM 1. *Let $f(z_1,z_2)$ be of L-order ρ^* and lower L-order λ^*. Then*

$$\lim_{r_1,r_2\to\infty}\ {\sup \atop \inf}\ \frac{\log\phi(r_1,r_2)}{\log\log(r_1 r_2)}=\frac{\rho^*}{\lambda^*},$$

where $\phi(r_1,r_2)$ stands for the logarithms of any of the means defined in (2.1)—(2.6).

THEOREM 2. *Let $f(z_1,z_2)$ be of L-order ρ^*, and also let*

(3.1) $$\lim_{r_1,r_2\to\infty}\ {\sup \atop \inf}\ \left\{\frac{I_\delta(r_1,r_2)}{m_{\delta,\lambda,\mu}(r_1,r_2)}\right\}^{\frac{1}{\log\log(r_1 r_2)}}=\frac{H_{\delta,\lambda,\mu}}{h_{\delta,\lambda,\mu}}.$$

Then

(3.2) $$H_{\delta,\lambda,\mu}=e^{\rho^*-1},$$

provided

$$h^2_{\delta,\lambda,\mu}<H_{\delta,\lambda,\mu}.$$

If we replace $m_{\delta,\lambda,\mu}(r_1,r_2)$ in (3.1) by $m^*_{\delta,\lambda,\mu}(r_1,r_2)$, then we have a different result; namely

THEOREM 3. *Let $f(z_1,z_2)$ be of L-order ρ^*, and also let*

$$\lim_{r_1,r_2\to\infty}\ {\sup \atop \inf}\ \left\{\frac{I_\delta(r_1,r_2)}{m^*_{\delta,\lambda,\mu}(r_1,r_2)}\right\}^{\frac{1}{\log\log(r_1 r_2)}}=\frac{L_{\delta,\lambda,\mu}}{l_{\delta,\lambda,\mu}}.$$

Then

$$L_{\delta,\lambda,\mu}=e^{\rho^*},$$

provided

$$l^2_{\delta,\lambda,\mu}<L_{\delta,\lambda,\mu}.$$

To state the last theorem of this paper more precisely, let us consider a slowly growing real function $\varphi(r_1,r_2)$ of two variables, such that

(i) $\varphi(r_1, r_2)>0$ and continuous for $r_1 \geq r_1^0$, $r_2 \geq r_2^0$,

(ii) for every $\alpha, \beta>1$, $\varphi(r_1^\alpha, r_2^\beta) \sim \varphi(r_1, r_2)$, as r_1 or r_2 or r_1 and r_2 tend to infinity;

(iii) for $A>0$

$$\int^{r_1} (\log x_1)^A \varphi(x_1, r_2) \frac{dx_1}{x_1} \sim \frac{(\log r_1)^{A+1} \varphi(r_1, r_2)}{A+1},$$

$$\int^{r_2} (\log x_2)^A \varphi(r_1, x_2) \frac{dx_2}{x_2} \sim \frac{(\log r_2)^{A+1} \varphi(r_1, r_2)}{A+1}.$$

Now, we have the following theorem which gives certain comperative growth relations of the means defined in (2.4) and (2.6) when $\lambda=\mu$. It may be remarked here that the results so obtained may not be generalized in terms of $\log g^*_{\lambda,\mu}(r_1, r_2)$ when $\lambda \neq \mu$.

THEOREM 4. *Let $f(z_1, z_2)$ be of finite L-order ρ^*. Further, let p, q $(0<q \leq p<\infty)$ and $c, d (0<d \leq c<\infty)$ be defined by*

$$\lim_{r_1, r_2 \to \infty} \frac{\sup}{\inf} \left\{ \frac{\log g^*_{\lambda,\lambda}(r_1, r_2)}{[(\log r_1)^{\rho^*} + (\log r_2)^{\rho^*}] \phi(r_1, r_2)} \right\} = \frac{p}{q}, \tag{3.3}$$

and

$$\lim_{r_1, r_2 \to \infty} \frac{\sup}{\inf} \left\{ \frac{\log G(r_1, r_2)}{[(\log r_1)^{\rho^*} + (\log r_2)^{\rho^*}] \varphi(r_1, r_2)} \right\} = \frac{c}{d}.$$

Then

$$\frac{\lambda+1}{\rho^*+\lambda+1} d \leq q \leq p \leq \frac{\lambda+1}{\rho^*+\lambda+1} c, \tag{3.4}$$

$$c \leq \left(\frac{\rho^*+2\lambda+2}{2\lambda+2} \right)^2 \left(\frac{\rho^*+2\lambda+2}{\rho^*} \right)^{\frac{\rho^*}{\lambda+1}} p. \tag{3.5}$$

COROLLARY. *Under the hypothesis of the theorem, if $c=d$, then*

$$p=q=\frac{\lambda+1}{\rho^*+\lambda+1} c.$$

REMARK. The authors are of the view that to establish an analogue of theorem 4, when $g^*_{\lambda,\lambda}(r_1, r_2)$ in (3.3) is replaced by $g_{\lambda,\lambda}(r_1, r_2)$, is an extreemly hard job.

4. Proof of Theorem 1

We have, for[2] $R_i > r_i \geq r_i^0$ $(i=1,2)$, the following inequalities:

$$(4.1)\qquad I_\delta(r_1,r_2) \leq M(r_1,r_2) \leq \frac{(R_1+r_1)^{\frac{1}{\delta}}(R_2+r_2)^{\frac{1}{\delta}}}{(R_1-r_1)^{\frac{1}{\delta}}(R_2-r_2)^{\frac{1}{\delta}}}\, I_\delta(R_1,R_2),$$

$$(4.2)\qquad m_{\delta,\lambda,\mu}(r_1,r_2) \leq \frac{I_\delta(r_1,r_2)}{(\lambda+1)(\mu+1)} \leq \frac{R_1^{\lambda+1}R_2^{\mu+1}}{(R_1^{\lambda+1}-r_1^{\lambda+1})(R_2^{\mu+1}-r_2^{\mu+1})}\, m_{\delta,\lambda,\mu}(R_1,R_2),$$

$$(4.3)\qquad m^*_{\delta,\lambda,\mu}(r_1,r_2) \leq \frac{I_\delta(r_1,r_2)}{(\lambda+1)(\mu+1)} \leq \frac{(\log R_1)^{\lambda+1}(\log R_2)^{\mu+1}}{((\log R_1)^{\lambda+1}-(\log r_1)^{\lambda+1})((\log R_2)^{\mu+1}-(\log r_2)^{\mu+1})}\, m^*_{\delta,\lambda,\mu}(R_1,R_2),$$

$$(4.4)\qquad \log G(r_1,r_2) \leq \log^+ M(r_1,r_2) \leq \frac{(R_1+r_1)(R_2+r_2)}{(R_1-r_1)(R_2-r_2)} \log G(R_1,R_2),$$

$$(4.5)\qquad \log g_{\lambda,\mu}(r_1,r_2) \leq \log G(r_1,r_2) \leq \frac{R_1^{\lambda+1}R_2^{\mu+1}}{(R_1^{\lambda+1}-r_1^{\lambda+1})(R_2^{\mu+1}-r_2^{\mu+1})} \log g_{\lambda,\mu}(R_1,R_2)$$

and

$$(4.6)\qquad \log g^*_{\lambda,\mu}(r_1,r_2) \leq \log G(r_1,r_2) \leq \frac{(\log R_1)^{\lambda+1}(\log R_2)^{\mu+1}}{((\log R_1)^{\lambda+1}-(\log r_1)^{\lambda+1})((\log R_2)^{\mu+1}-(\log r_2)^{\mu+1}))} \log g^*_{\lambda,\mu}(R_1,R_2)$$

The inequalities (4.1), (4.2), (4.4) and (4.5) have been obtained in [1], [7], [6] whereas the proofs of (4.3) and (4.6) will follow on the lines of those of (4.2) and (4.5) respectively.

Now, on putting $R_i=2r_i$ in (4.1), (4.2), (4.4), (4.5) and $R_i=r_i^2$ in (4.3) and (4.6), the theorem follows completely.

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2) We take $r_i^0 \geq 1$ for the inequalities in (4.3) and (4.6) whereas $r_i^0 \geq 0$ for the others.

5. Proofs of Theorem 2 and 3

We have (see [1], p.52)

$$\frac{\partial^2}{\partial r_1 \partial r_2}\{\log(r_1^{\lambda+1} r_2^{\mu+1} m_{\delta,\lambda,\mu}(r_1,r_2))\}$$

(5.1)
$$= \frac{1}{[r_1^{\lambda+1} r_2^{\mu+1} m_{\delta,\lambda,\mu}(r_1,r_2)]^2}\Big\{(r_1^{\lambda} r_2^{\mu} I_\delta(r_1,r_2))(r_1^{\lambda+1} r_1^{\mu+1} m_{\delta,\lambda,\mu}(r_1,r_2))$$
$$-[r_2^{\mu}\int_0^{r_1} x_1^{\lambda} I_\delta(x_1,r_2)dx_1]\,[r_1^{\lambda}\int_0^{r_2} x_2^{\mu} I_\delta(r_1,x_2)dx_2]\Big\}$$

(5.2)
$$\leq \frac{I_\delta(r_1,r_2)}{m_{\delta,\lambda,\mu}(r_1,r_2)}\,\frac{1}{r_1 r_2}.$$

Now, integrating both the sides of the inequality in (5.2), we get

(5.3)
$$\log(r_1^{\lambda+1} r_2^{\mu+1} m_{\delta,\lambda,\mu}(r_1,r_2)) \leq \int_0^{r_1}\int_0^{r_2} \frac{I_\delta(x_1,x_2)}{m_{\delta,\lambda,\mu}(x_1,x_2)}\,\frac{dx_1}{x_1}\,\frac{dx_2}{x_2}.$$

Let $H_{\delta,\lambda,\mu}<\infty$. Then, for $\varepsilon>0$, (5.3) gives

$$\log\,(r_1^{\lambda+1} r_2^{\mu+1} m_{\delta,\lambda,\mu}(r_1,r_2)) \leq \int_{r_1^0}^{r_1}\int_{r_2^0}^{r_2} (H_{\delta,\lambda,\mu}+\varepsilon)^{\log\,\log(x_1 x_2)}\frac{dx_1}{x_1}\,\frac{dx_2}{x_2}+o(1)$$
$$< \frac{(H_{\delta,\lambda,\mu}+\varepsilon)^{\log\,\log(r_1 r_2)}\log(r_1 r_2)}{[\log(H_{\delta,\lambda,\mu}+\varepsilon)]^2}+o(1),$$

which implies

(5.4)
$$\rho^* \leq \log\, H_{\delta,\lambda,\mu}+1.$$

Next, integrating (5.1), we get

$$\log\{(r_1^{\,2})^{\lambda+1}(r_2^{\,2})^{\mu+1} m_{\delta,\lambda,\mu}(r_1^{\,2},r_2^{\,2})\}$$

(5.5)
$$> \Big\{\int_{r_1}^{r_1^2}\int_{r_2}^{r_2^2} \frac{I_\delta(x_1,x_2)}{m_{\delta,\lambda,\mu}(x_1,x_2)}\,\frac{dx_1}{x_1}\,\frac{dx_2}{x_2}$$
$$-\int_{r_1}^{r_1^2}\int_{r_2}^{r_2^2}\Big[\frac{I_\delta(x_1,x_2)}{m_{\delta,\lambda,\mu}(x_1,x_2)}\Big]^2\frac{dx_1}{x_1}\,\frac{dx_2}{x_2}\Big\}$$

Since $\{r_1^{\lambda+1} r_2^{\mu+1} I_\delta(r_1,r_2)\}$ is a convex function of $\{r_1^{\lambda+1} r_2^{\mu+1} m_{\delta,\lambda,\mu}(r_1,r_2)\}$ when one of the variable r_1 is fixed and the other variable r_2 increases, vice-versa or both increase (cf [9], p.194).

Hence, using (3.1) in (5.5), for certain sequences of r_1 and r_2(say) $\{r_{1,i}\}$ and $\{r_{2,j}\}$ ($r_{1,i}\to\infty$ with i and $r_{2,j}\to\infty$ with j), we obtain

$$\log\{(r_1^{\,2})^{\lambda+1}(r_2^{\,2})^{\mu+1}m_{\delta,\lambda,\mu}(r_1^{\,2},r_2^{\,2})\}$$

$$>\left\{\frac{I_\delta(r_1,r_2)}{m_{\delta,\lambda,\mu}(r_1,r_2)}-\left[\frac{I_\delta(r_1,r_2)}{m_{\delta,\lambda,\mu}(r_1,r_2)}\right]^2\right\}\log(r_1r_2)$$

$$>\{(H_{\delta,\lambda,\mu}-\varepsilon)^{\log\log(r_1r_2)}-(h_{\delta,\lambda,\mu}+\varepsilon)^{2\log\log(r_1r_2)}\}\log(r_1r_2)$$

$$\sim(H_{\delta,\lambda,\mu}-\varepsilon)^{\log\log(r_1r_2)}\log(r_1r_2),$$

since $h^2_{\delta,\lambda,\mu}<H_{\delta,\lambda,\mu}$, and the result follows in view of (5.4).

Proof of theorem 3 follows on the lines of theorem 2, and is omitted.

6. Proof of Theorem 4

Let $\eta>1$. Then, from (2.6), we have

$$(6.1)\quad \log g^*_{\lambda,\lambda}\left(r_1^{\eta^{\frac{1}{\rho^*}}},r_2^{\eta^{\frac{1}{\rho^*}}}\right)=(\lambda+1)^2\eta^{-\frac{2(\lambda+1)}{\rho^*}}(\log r_1)^{-(\lambda+1)}(\log r_2)^{-(\lambda+1)}$$

$$\int_1^{r_1^{\eta^{\frac{1}{\rho^*}}}}\int_1^{r_2^{\eta^{\frac{1}{\rho^*}}}}(\log x_1)^\lambda(\log x_2)^\lambda\log G(x_1,x_2)\frac{dx_1}{x_1}\frac{dx_2}{x_2}$$

$$<A(\log r_1)^{-(\lambda+1)}(\log r_2)^{-(\lambda+1)}+\eta^{-\frac{2(\lambda+1)}{\rho^*}}(\log r_1^0)^{\lambda+1}$$

$$\times(\log r_1)^{-(\lambda+1)}\left\{\log G(r_1^0,r_2)+\left(\eta^{\frac{\lambda+1}{\rho^*}}-1\right)\log G\left(r_1^0,r_2^{\eta^{\frac{1}{\rho^*}}}\right)\right\}$$

$$+\eta^{-\frac{2(\lambda+1)}{\rho^*}}(\log r_2^0)^{(\lambda+1)}(\log r_2)^{-(\lambda+1)}\{\log G(r_1,r_2^0)$$

$$+(\eta^{\frac{\lambda+1}{\rho^*}}-1)\log G(r_1^{\eta^{\frac{1}{\rho^*}}},r_2^0)\}$$

$$+(\lambda+1)^2(c+\varepsilon)\eta^{-\frac{2(\lambda+1)}{\rho^*}}(\log r_1)^{-(\lambda+1)}(\log r_2)^{-(\lambda+1)}$$

$$\times\int_{r_1^0}^{r_1}\int_{r_2^0}^{r_2}(\log x_1)^\lambda(\log x_2)^\lambda\{(\log x_1)^{\rho^*}+(\log x_2)^{\rho^*}\}$$

$$\times\varphi(x_1,x_2)\frac{dx_1}{x_1}\frac{dx_2}{x_2}$$

$$+(\lambda+1)\eta^{-\frac{2(\lambda+1)}{\rho^*}}(\eta^{\frac{\lambda+1}{\rho^*}}-1)\{(\log r_1)^{-(\lambda+1)}$$

$$\times\int_{r_1^0}^{r_1}(\log x_1)^{\lambda}\log G(x_1, r_2^{\eta^{\frac{1}{\rho^*}}})\frac{dx_1}{x_1}$$

$$+(\log r_2)^{-(\lambda+1)}\int_{r_2^0}^{r_2}(\log x_2)^{\lambda}\log G(r_1^{\eta^{\frac{1}{\rho^*}}}, x_2)\frac{dx_2}{x_2}$$

$$+\eta^{-\frac{2(\lambda+1)}{\rho^*}}(\eta^{\frac{\lambda+1}{\rho^*}}-1)^2\log G(r_1^{\eta^{\frac{1}{\rho^*}}}, r_2^{\eta^{\frac{1}{\rho^*}}})\}$$

$$\sim A(\log r_1)^{-(\lambda+1)}(\log r_2)^{-(\lambda+1)}+\eta^{-\frac{2(\lambda+1)}{\rho^*}}(\log r_1^0)^{\lambda+1}(\log r_1)^{-(\lambda+1)}$$

$$\times\{\log G(r_1^0, r_2)+(\eta^{\frac{\lambda+1}{\rho^*}}-1)\log G(r_1^0, r_2^{\eta^{\frac{1}{\rho^*}}})\}$$

$$+\eta^{-\frac{2(\lambda+1)}{\rho^*}}(\log r_2^0)^{\lambda+1}(\log r2)^{-(\lambda+1)}\{\log G(r_1, r_2^0)$$

$$+(\eta^{\frac{\lambda+1}{\rho^*}}-1)\log G(r_1^{\eta^{\frac{1}{\rho^*}}}, r_1^0)\}+(\lambda+1)(c+\varepsilon)(p^*+\lambda+1)^{-1}\eta^{-\frac{2(\lambda+1)}{\rho^*}}$$

$$\{(\log r_1)^{\rho^*}+(\log r_2)^{\rho^*}\}\varphi(r_1, r_2)$$

$$+(c+\varepsilon)\eta^{-\frac{2(\lambda+1)}{\rho^*}}(\eta^{\frac{\lambda+1}{\rho^*}}-1)\{[(\log r_1)^{\rho^*}+\eta(\log r_2)^{\rho^*}]\varphi(r_1, r_2^{\eta^{\frac{1}{\rho^*}}})$$

$$+[\eta(\log r_1)^{\rho^*}+(\log r_2)^{\rho^*}]\varphi(r_1^{\eta^{\frac{1}{\rho^*}}}, r_2)\}$$

$$+\eta^{-\frac{2(\lambda+1)}{\rho^*}}(\eta^{\frac{\lambda+1}{\rho^*}}-1)^2\log G(r_1^{\eta^{\frac{1}{\rho^*}}}, r_2^{\eta^{\frac{1}{\rho^*}}}).$$

On dividing both sides of the above by $\{\eta[(\log r_1)^{\rho^*}+(\log r_2)^{\rho^*}]\varphi(r_1^{\eta^{\frac{1}{\rho^*}}}, r2^{\eta^{\frac{1}{\rho^*}}})\}$, and taking the limlt, we get

$$p\leq(\lambda+1)(\rho^*+\lambda+1)^{-1}\eta^{-\frac{2(\lambda+1)}{\rho^*}-1}c+\eta^{-\frac{2(\lambda+1)}{\rho^*}}(\eta^{\frac{\lambda+1}{\rho^*}}-1)c+\eta^{-\frac{2(\lambda+1)}{\rho^*}}(\eta^{\frac{\lambda+1}{\rho^*}}-1)^2c.$$

Letting $\eta\to 1$, η being arbitrary, the extreme right-hand inequality in (3.4) follows.

Further, from (6.1), we have

$$\text{(6.2)}\qquad \log g^*_{\lambda,\lambda}(r_1^{\eta^{\frac{1}{\rho^*}}}, r_2^{\eta^{\frac{1}{\rho^*}}})>(\lambda+1)^2(d-\varepsilon)\eta^{-\frac{2(\lambda+1)}{\rho^*}}$$

$$(\log r_1)^{-(\lambda+1)}(\log r_2)^{-(\lambda+1)}\int_{r_1^0}^{r_1}\int_{r_2^0}^{r_2}(\log x_1)^{\lambda}(\log x_2)^{\lambda}$$

$$\times\{(\log x_1)^{\rho^*}+(\log x^2)^{\rho^*}\}\varphi(x_1,x_2)\,\frac{dx_1}{x_1}\,\frac{dx_2}{x_2}$$

$$+(\lambda+1)(d-\varepsilon)\eta^{-\frac{2(\lambda+1)}{\rho^*}}(\eta^{\frac{\lambda+1}{\rho^*}}-1)\Big\{(\log r_1)^{-(\lambda+1)}$$

$$\times\int_{r_1^0}^{r_1}(\log x_1)^{\lambda}\{(\log x_1)^{\rho^*}+(\log r_2)^{\rho^*}\}\varphi(x_1,r_2)\,\frac{dx_1}{x_1}$$

$$+(\log r_2)^{-(\lambda+1)}\int_{r_2^0}^{r_2}(\log x_2)^{\lambda}\{(\log r_1)^{\rho^*}+(\log x_2)^{\rho^*}\}\varphi(r_1,r_2)\,\frac{dx_2}{x_2}\Big\}$$

$$+\eta^{-\frac{2(\lambda+1)}{\rho^*}}(\eta^{\frac{\lambda+1}{\rho^*}}-1)^2\log G(r_1,r_2).$$

$$\sim(d-\varepsilon)(\lambda+1)(\rho^*+\lambda+1)^{-1}\eta^{-\frac{2(\lambda+1)}{\rho^*}}\{(\log r_1)^{\rho^*}+(\log r_2)^{\rho^*}\}\varphi(r_1,r_2)$$

$$+(d-\varepsilon)(\rho^*+2\lambda+2)(\rho^*+\lambda+1)^{-1}\eta^{-\frac{2(\lambda+1)}{\rho^*}}(\eta^{\frac{\lambda+1}{\rho^*}}-1)$$

$$\times\{(\log r_1)^{\rho^*}+(\log r_2)^{\rho^*}\}\varphi(r_1,r_2)+\eta^{-\frac{2(\lambda+1)}{\rho^*}}(\eta^{\frac{\lambda+1}{\rho^*}}-1)^2\log G(r_1,r_2),$$

and hence the left hand inequality in (3.4) also follows.

To establish the last inequality in the theorem, we note, from (6.2), that

$$(\eta^{\frac{\lambda+1}{\rho^*}}-1)^2c\leq\eta\cdot\eta^{\frac{2(\lambda+1)}{\rho^*}}p-d(\lambda+1)(\rho^*+\lambda+1)^{-1}$$

$$\{\eta^{-\frac{2(\lambda+1)}{\rho^*}}+\eta^{-\frac{2(\lambda+1)}{\rho^*}}(\eta^{\frac{p^*+\lambda+1}{\rho^*}}+\eta^{-\frac{\lambda+1}{\rho^*}}-2)\}$$

which implies

$$c\leq(\eta^{\frac{\lambda+1}{\rho^*}}-1)^{-2}\eta^{\frac{\rho^*+2\lambda+2}{\rho^*}}p, \tag{6.3}$$

for all values of $\eta>1$. But the right hand side of (6.3) attains minimum when η is given by

$$\eta^{\frac{\lambda+1}{\rho^*}}=\frac{(\rho^*+2\lambda+2)}{\rho^*}.$$

Substituting this value of η in (6.3), the remaining part of the theorem follows.

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