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SOME GENERALISATIONS OF METACOMPACT SPACES

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1. Introduction

As defined by Arens and Dugundji [2], a space X is called metacompact if each open covering of X admits of a point-finite open refinement. The purpose of the present paper is to study some generalisations of this concept, namely $\mathfrak{M}$-metacompactness and almost $\mathfrak{M}$-metacompactness, $\mathfrak{M}$ being an infinite cardinal. The first one of these is a generalisation of metacompactness as also that of $\mathfrak{M}$-paracompactness introduced by Morita [15]. The latter one appears as a generalization of almost $\mathfrak{M}$-paracompactness introduced by Singal and Arya [18].

Let $\mathfrak{M}$ be an infinite cardinal. A space X is called $\mathfrak{M}$-metacompact if every open covering of X of cardinality $\leq\mathfrak{M}$ admits a point finite open refinement. If $\mathscr{U}$ and $\mathscr{V}$ are collections of subsets of X such that each member of $\mathscr{V}$ is contained in some member of $\mathscr{U}$, then $\mathscr{V}$ is called a weak refinement of $\mathscr{U}$. If further $\bigcup\{U : U\in\mathscr{U}\}=\bigcup\{V : V\in\mathscr{V}\}$, then $\mathscr{V}$ is called a refinement of $\mathscr{U}$. A space X is called almost $\mathfrak{M}$-metacompact if, for every open covering of X of cardinality $\leq\mathfrak{M}$ there is a point-finite open weak refinement whose closures cover X. Clearly, every almost $\mathfrak{M}$-paracompact as also every $\mathfrak{M}$-metacompact space is almost $\mathfrak{M}$-metacompact. A space is metacompact (almost metacompact) if it is $\mathfrak{M}$-metacompact (almost $\mathfrak{M}$-metacompact) for every infinite cardinal $\mathfrak{M}$. For a topological space X with an open base of cardinality $\leq\mathfrak{M}$, $\mathfrak{M}$-metacompactness (almost $\mathfrak{M}$-metacompactness) is equivalent to metacompactness (almost metacompactness). Since for a screenable space (that is, a space in which every open covering has a σ-mutually disjoint open refinement) as well as for a metaLindelöf space (that is, a space in which every open covering has a point-countable open refinement) metacompactness is equivalent to countable metacompactness [5], it follows that for such spaces metacompactness is equivalent to $\mathfrak{M}$-metacompactness for any infinite cardinal $\mathfrak{M}$.

In section 2 we obtain some new results for countable metacompactness. Sections 3, 4 and 5 deal with $\mathfrak{M}$-metacompactness and in the last section almost $\mathfrak{M}$-metacompactness has been discussed.

Throughout X will denote a topological space and $\mathfrak{M}$ will denote an infinite cardinal. N denotes the set of natural numbers. For any point set Λ, $|\Lambda|$ will denote the cardinality of Λ.

2. Countably metacompact spaces

For our first theorem we need the following result due to Hayashi [6] which characterizes countably metacompact spaces.

THEOREM 2.1. [Hayashi, 6]. *A space X is countably metacompact if and only if for every decreasing sequence $\{F_i\}$ of closed sets with empty intersection there is a sequence $\{G_i\}$ of open sets with empty intersection such that $G_i \supset F_i$ for each i.*

THEOREM 2.2. *Every closed, continuous image of a countably metacompact space X is countably metacompact.*

PROOF. Let f be a closed continuous mapping of a countably metacompact space X onto a space Y. Let $\{F_i\}$ be a decreasing sequence of closed subsets of Y with empty intersection. Then, since f is continuous, $\{f^{-1}(F_i)\}$ is a decreasing sequence of closed sets with empty intersection. Since X is countably metacompact, there is a sequence $\{G_i\}$ of open subsets of X such that $\bigcap_{i=1}^{\infty} G_i=\phi$ and $f^{-1}(F_i)\subset G_i$ for all i. Now, if $(G_i)_0$ be the union of all sets $f^{-1}(y)$ which are contained in G_i, then $(G_i)_0$ is an open inverse set because f is closed and continuous. Also, $f^{-1}(F_i)\subset (G_i)_0 \subset G_i$ and $\bigcap_{i=1}^{\infty} (G_i)_0=\phi$. Then $F_i=f((G_i)_0)$ for all i. Also $\bigcap_{i=1}^{\infty} f((G_i)_0)=\phi$, for if $y\in f((G_i)_0)$ for all i, then $f^{-1}(y)\subset (G_i)_0$ for all i, since $(G_i)_0$ is an inverse set. Hence $f^{-1}(y)\subset G_i$ for all i, that is, $f^{-1}(y)\subset \bigcap_{i=1}^{\infty} G_i$ and thus $\bigcap_{i=1}^{\infty} (G_i)_0 \neq\phi$, which is a contradiction. Also, since f is closed and continuous, $f((G_i)_0)$ is an open set. Thus $\{f((G_i)_0)\}$ is a sequence of open sets with empty intersection such that $F_i \subset f((G_i)_0)$ for all i. Hence Y is countably metacompact in view of Theorem 2.1.

THEOREM 2.3. *Disjoint topological sum of countably metacompact spaces is countably metacompact.*

PROOF. Let $\{X_\alpha : \alpha\in\Lambda\}$ be a disjoint family of countably metacompact spaces. Let X denote the disjoint topological sum of this family.
Let $\mathscr{U}=\{U_n : n\in N\}$ be a countable open covering of X. For each $\alpha\in\Lambda$, $\{U_n \cap X_\alpha : n\in N\}$ is a countable open covering of X_α. Let $\mathscr{V}_\alpha$ be a point-finite open

refinement of $\{U_n \cap X_\alpha : n \in N\}$. Since the family $\{X_\alpha : \alpha \in \Lambda\}$ is a disjoint family and each $\mathscr{V}_\alpha$ is point-finite, we conclude that $\bigcup_{\alpha\in\Lambda} \mathscr{V}_\alpha$ is a point-finite open refinement of $\mathscr{U}$. Hence X is countably metacompact.

THEOREM 2.4. *If $\{F_\alpha : \alpha \in \Lambda\}$ is a locally finite closed covering of X such that each F_α is countably metacompact, then X is countably metacompact.*

PROOF. For each $\alpha \in \Lambda$, let X_α be a homeomorphic copy of F_α and let f_α denote a homeomorphism from X_α to F_α. Let X^* denote the disjoint topological sum of the X_α 's. Then X^* is countably metacompact in view of Theorem 2.3. Define the natural map $f : X^* \to X$ as $f(x^*) = f_\alpha(x^*)$ if $x^* \in X_\alpha$. It can be easily verified that f is a closed, continuous mapping. Hence X is countably metacompact by Theorem 2.2.

Hodel [10], Singal and Arya ([19], [20], [21]) have obtained several sum theorems for a topological property $\mathscr{P}$ which is closed hereditary and which satisfies the following property:

'If $\{F_\alpha : \alpha \in \Lambda\}$ is a locally finite closed covering of X such that each F_α has the property $\mathscr{P}$, then X has the property $\mathscr{P}$.'

Thus in view of Theorem 2.4 and the fact that every closed subset of a countably metacompact space is countably metacompact, we obtain the following sum theorems.

THEOREM 2.5. *If $\mathscr{V}$ is an order locally finite open covering of X such that the closure of each member of $\mathscr{V}$ is countably metacompact, then X is countably metacompact.*

COROLLARY 2.1. *If $\mathscr{V}$ be a σ-locally finite open covering of X such that the closure of each member of $\mathscr{V}$ is countably metacompact, then X is countably metacompact.*

THEOREM 2.6. *If X is regular and $\mathscr{V}$ is an order locally finite open covering of X such that for each $V \in \mathscr{V}$, V is countably metacompact and* $\mathrm{Fr}(V)$ *is compact, then X is countably metacompact.*

COROLLARY 2.2. *If X is regular and $\mathscr{V}$ is a σ-locally finite open covering of X such that for each $V \in \mathscr{V}$, V is countably metacompact and* $\mathrm{Fr}(V)$ *is compact, then X is countably metacompact.*

THEOREM 2.7. *Let $\mathscr{V}$ be a σ-locally finite elementary covering of X such that each $V \in \mathscr{V}$ is countably metacompact. Then X is countably metacompact.*

THEOREM 2.8. *Let $\mathscr{V}$ be a locally finite open covering of a regular space X such that for each $V\in\mathscr{V}$, V is countably metacompact and* $\mathrm{Fr}(V)$ *is Lindelöf, then X is countably metacompact.*

THEOREM 2.9. *Let $\mathscr{V}$ be a normal open covering of a normal space X such that each member of $\mathscr{V}$ is countably metacompact. Then X is countably metacompact.*

PROOF. By Theorem 1.2 in [15], $\mathscr{V}$ admits of a locally finite (and hence point-finite) open refinement. Therefore the result follows in view of Remark 3.3.

THEOREM 2.10. *Let $\mathscr{V}$ be a σ-locally finite open covering of a normal space X such that each $V\in\mathscr{V}$ is an F_σ-subset of X. Then X is countably metacompact if each $V\in\mathscr{V}$ is countably metacompact.*

PROOF. The theorem is an easy consequence of Theorem 2.7 and the fact that every open F_σ-subset of a normal space is elementary.

THEOREM 2.11. *Every space which contains a proper non-empty regularly closed set is countably metacompact if and only if every regularly closed subset of X is countably metacompact.*

COROLLARY 2.3. *A weakly regular space X is countably metacompact if and only if every proper regularly closed subset of X is countably metacompact.*

COROLLARY 2.4. *A semi-regular space X is countably meta compact if and only if every proper regularly closed subset of X is countably metacompact.*

3. Characterizations and conditions implying $\mathfrak{M}$-metacompactness

THEOREM 3.1. *A space X is $\mathfrak{M}$-metacompact if and only if it is countably metacompact and each open covering of X of cardinality $\leq\mathfrak{M}$ admits of a σ-point-finite open refinement.*

PROOF. The 'only if' part is obvious. We shall prove the 'if' part. Let $\mathscr{U}=\{U_\alpha : \alpha\in\Lambda\}$ be any open covering of X with $|\Lambda|\leq\mathfrak{M}$. By hypothesis there exists a σ-point-finite open refinement $\mathscr{V}=\bigcup_{i=1}^{\infty}\mathscr{V}_i$ of $\mathscr{U}$ where each $\mathscr{V}_i=\{V_{\beta,i} : \beta\in\Lambda_i\}$ is a point finite collection. For each $i\in N$, let $V_i=\bigcup\{V_{\beta,i} : \beta\in\Lambda_i\}$. Then $\{V_i : i\in N\}$ is a countable open covering of X and since X is countably metacompact, there exists a point-finite open refinement $\{W_i : i\in N\}$ of $\{V_i : i\in N\}$ such that $W_i\subset V_i$ for each i. Then, $\{W_i\cap V_{\beta,i} : \beta\in\Lambda_i,\ i\in N\}$ is a point-finite open refinement of $\mathscr{U}$ and hence X is $\mathfrak{M}$-metacompact.

Arens and Dugundji [2] proved that a T_1 space is compact if and only if it is countably compact and metacompact. With essentially the same argument we obtain the following.

THEOREM 3.2. *A T_1 space X is $\mathfrak{M}$-compact if and only if it is countably compact and $\mathfrak{M}$-metacompact.*

COROLLARY 3.1. *A T_1 space X is $\mathfrak{M}$-compact if and only if it is countably compact and $\mathfrak{M}$-paracompact.*

REMARK 3.1. Corollary 2.1 was proved by Morita [15] with the assumption that X is normal.

THEOREM 3.3. *Every collectionwise normal $\mathfrak{M}$-metacompact space is $\mathfrak{M}$-paracompact.*

PROOF. Follows easily from Theorem 1 of Michael [14].

COROLLARY 3.2. *In a collectionwise normal space, $\mathfrak{M}$-metacompactness is equivalent to $\mathfrak{M}$-paracompactness.*

DEFINITION 3.1. [Krajewski, 14] A space X is said to be $\mathfrak{M}$-*expandable* if for every locally finite collection $\{H_\alpha : \alpha \in \Lambda\}$ of subsets of X with $|\Lambda| \leq \mathfrak{M}$ there is a locally finite collection of open subsets $\{G_\alpha : \alpha \in \Lambda\}$ such that $F_\alpha \subset G_\alpha$ for every $\alpha \in \Lambda$. X is *expandable* if it is $\mathfrak{M}$-expandable for every infinite cardinal $\mathfrak{M}$.

It is clear that X is $\mathfrak{M}$-expandable if and only if for every locally finite collection of closed subsets $\{F_\alpha : \alpha \in \Lambda\}$ with $|\Lambda| \leq \mathfrak{M}$ there exists a locally finite collection of open subsets $\{G_\alpha : \alpha \in \Lambda\}$ such that $F_\alpha \subset G_\alpha$ for each $\alpha \in \Lambda$.

It has been proved by Krajewski [14] that collectionwise normality in Theorem 3.2 can be replaced by $\mathfrak{M}$-expandability. In fact he has proved the following:

THEOREM 3.4. *X is $\mathfrak{M}$-paracompact if and only if X is $\mathfrak{M}$-metacompact and $\mathfrak{M}$-expandable.*

DEFINITION 3.2. [22] A space X is said to be $\mathfrak{M}$-*subparacompact* if every open covering of X of cardinality $\leq \mathfrak{M}$ admits of a σ-discrete closed refinement.

LEMMA 3.1. *Let X be a topological space in which every closed set is a G_δ-set (that is a countable intersection of open sets). Then every point-finite open covering of X has a σ-discrete closed refinement.*

Thus, we have the following:

THEOREM 3.5. *Every $\mathfrak{M}$-metacompact space in which every closed set is a G_δ-set is $\mathfrak{M}$-subparacompact.*

THEOREM 3.6. *A necessary condition for a space X to be $\mathfrak{M}$-metacompact is that for every locally finite family of closed sets $\{F_\alpha : \alpha \in \Lambda\}$ with $|\Lambda| \leq \mathfrak{M}$ there exists a point-finite family of open sets $\{G_\alpha : \alpha \in \Lambda\}$ such that $F_\alpha \subset G_\alpha$ for every $\alpha \in \Lambda$.*

PROOF. Let $\mathscr{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a locally finite family of closed subsets of X with $|\Lambda| \leq \mathfrak{M}$. Let Γ be the collection of all finite subsets of Λ. Then $|\Gamma| \leq \mathfrak{M}$. For each $\gamma \in \Gamma$, let

$$V_\gamma = X - \bigcup \{F_\alpha : \alpha \notin \gamma\}.$$

Then $\mathscr{V} = \{V_\gamma : \gamma \in \Gamma\}$ is an open cover of X of cardinality $\leq \mathfrak{M}$ such that each member of $\mathscr{V}$ intersects only finitely many members of $\mathscr{F}$. Since X is $\mathfrak{M}$-metacompact, there is a point-finite open refinement $\mathscr{W} = \{W_\beta : \beta \in \Delta\}$ of $\mathscr{V}$. For each $\alpha \in \Lambda$, let

$$G_\alpha = \mathrm{St}(F_\alpha, \mathscr{W}) = \bigcup \{W_\beta \in \mathscr{W} : W_\beta \cap F_\alpha \neq \phi\}.$$

Clearly, $\{G_\alpha : \alpha \in \Lambda\}$ is a collection of open subsets of X such that $F_\alpha \subset G_\alpha$ for each $\alpha \in \Lambda$. We shall prove that $\{G_\alpha : \alpha \in \Lambda\}$ is point-finite. Let $x \in X$. Then x belongs to only finitely many members of $\mathscr{W}$. Also $x \in G_\alpha$ if and only if $x \in W_\beta$ and $W_\beta \cap F_\alpha \neq \phi$ for some $\beta \in \Delta$. But W_β, being a subset of some V_γ, meets only finitely many F_α's. Thus $\{G_\alpha : \alpha \in \Lambda\}$ is point-finite.

DEFINITION 3.3. [Andenaes, 1]. A cover $\mathscr{U}$ of X is called *point-finite outside closed sets* if for each closed subset F of X and each point $x \in X - F$ there exist at most finitely many members of $\mathscr{U}$ which contain x and intersect F.

REMARK 3.2. A point-finite cover is obviously point-finite outside closed sets.

The following lemma has been proved in [1].

LEMMA 3.2. *Let $\mathscr{U}$ be an open cover of X which is point-finite outside closed sets. Then $\mathscr{U}$ has a point-finite subcover.*

In view of Lemma 3.2 and Remark 3.2, we obtain the following

THEOREM 3.7. *A space X is $\mathfrak{M}$-metacompact if and only if every open covering of X of cardinality $\leq \mathfrak{M}$ has an open refinement which is point-finite outside closed sets.*

THEOREM 3.8. *Let $\{G_\alpha : \alpha \in \Lambda\}$ be a family of subsets of X such that* {Int G_α : *$\alpha \in \Lambda$} forms a point-finite open covering of X. If each G_α is $\mathfrak{M}$-metacompact, then X is $\mathfrak{M}$-metacompact.*

PROOF. Let $\{U_\beta : \beta\in\Delta\}$ be any open covering of X with $|\Delta|\leq\mathfrak{M}$. Then, for each $\alpha\in\Lambda$, $\{U_\beta\cap G_\alpha : \beta\in\Delta\}$ is a relatively open covering of G_α of cardinality $\leq\mathfrak{M}$. Since G_α is $\mathfrak{M}$-metacompact, there exists a point-finite (in G_α) open (in G_α) covering $\{V_\gamma\cap G_\alpha : \gamma\in\Gamma_\alpha\}$ of G_α which refines $\{U_\beta\cap G_\alpha : \beta\in\Delta\}$ and each V_γ is open in X. Consider the family $\{V_\gamma\cap\text{Int } G_\alpha : \gamma\in\Gamma_\alpha,\ \alpha\in\Lambda\}$. This is a point-finite open refinement of $\{U_\beta : \beta\in\Delta\}$ and hence X is $\mathfrak{M}$-metacompact.

COROLLARY 3.3. *If $\{G_\alpha : \alpha\in\Lambda\}$ be a family of subsets of X sucht hat $\{\text{Int}G_\alpha : \alpha\in\Lambda\}$ is a point-finite covering of X, then X is metacompact (countably metacompact) if each G_α is metacompact (countably metacompact).*

THEOREM 3.9. *If $S\subset X$ where $S=\bigcup_{\alpha\in\Lambda} S_\alpha$ and each S_α is open in S and $\{S_\alpha : \alpha\in\Lambda\}$ is point-finite (in S), then S is $\mathfrak{M}$-metacompact if each S_α is $\mathfrak{M}$-metacompact.*

PROOF. Let $\{S\cap U_\beta : \beta\in\Delta\}$ be any relatively open covering of S with $|\Delta|\leq\mathfrak{M}$. Then for each $\alpha\in\Lambda$, $\{S_\alpha\cap U_\beta : \beta\in\Delta\}$ is a relatively open covering of S_α of cardinality $\leq\mathfrak{M}$. Since S_α is $\mathfrak{M}$-metacompact, there exists a point-finite (in S_α)family $\mathscr{V}_\alpha=\{V_{\alpha,\gamma} : \gamma\in\Gamma_\alpha\}$ of open subsets of S_α which refines $\{S_\alpha\cap U_\beta : \beta\in\Delta\}$ and hence also $\{U_\beta\cap S : \beta\in\Delta\}$. Now consider $\mathscr{V}=\{V_{\alpha,\gamma} : \gamma\in\Gamma_\alpha,\ \alpha\in\Lambda\}$. Then, each $V_{\alpha,\gamma}$ is an open subset of S. Also $\mathscr{V}$ is a point-finite open (in S) refinement of $\{S\cap U_\beta : \beta\in\Delta\}$ and thus S is $\mathfrak{M}$-metacompact.

COROLLARY 3.4. *If each member of a point-finite open covering of a space is $\mathfrak{M}$-metacompact, then the space is $\mathfrak{M}$-metacompact.*

COROLLARY 3.5. *Disjoint topological sum of $\mathfrak{M}$-metacompact spaces is $\mathfrak{M}$-metacompact.*

REMARK 3.3. The results of Corollary 3.4 and of Theorem 3.9 remain valid if the word '$\mathfrak{M}$-metacompact' be replaced by 'metacompact' or by 'countably metacompact'.

THEOREM 3.10. *Let $X=A\cup B$ where A and B are closed subsets of X. Then X is $\mathfrak{M}$-metacompact if A and B are both $\mathfrak{M}$-metacompact.*

PROOF. Let $\mathscr{U}=\{U_\alpha : \alpha\in\Lambda\}$ be an open covering of X of cardinality $\leq\mathfrak{M}$. Then $\{U_\alpha\cap A : \alpha\in\Lambda\}$ is an open (in A) covering of A of cardinality $\leq\mathfrak{M}$. Since A is $\mathfrak{M}$-metacompact there exists a point-finite open (in A) collection $\{V_\alpha : \alpha\in\Lambda\}$ covering A such that $V_\alpha\subset U_\alpha\cap A$ for all $\alpha\in\Lambda$. For $\alpha\in\Lambda$, let

$$U_\alpha^*=U_\alpha\cap[X-(A-V_\alpha)]$$

and let $\mathscr{U}^*=\{U_\alpha^*:\alpha\in\Lambda\}$. Now the collection $\mathscr{U}^*$ is an open cover of X such that $U_\alpha^*\subset U_\alpha$ for all $\alpha\in\Lambda$ and every point of A belongs to at most finitely many members of $\mathscr{U}^*$.

Again, $\{U_\alpha^*\cap B:\alpha\in\Lambda\}$ is an open (in B) covering of B of cardinality $\leq\mathfrak{M}$. Therefore there exist a point-finite open (in B) collection $\{W_\alpha:\alpha\in\Lambda\}$ covering B such that $W_\alpha\subset U_\alpha^*\cap B$ for all $\alpha\in\Lambda$. For each $\alpha\in\Lambda$, let

$$U_\alpha^{**}=U_\alpha^*\cap[X-(B-W_\alpha)]$$

and let $\mathscr{U}^{**}=\{U_\alpha^{**}:\alpha\in\Lambda\}$. Now the collection $\mathscr{U}^{**}$ is a point-finite open refinement of $\mathscr{U}$. Hence X is $\mathfrak{M}$-metacompact.

As a consequence of the above theorem we obtain following interesting results.

THEOREM 3.11. *Every space which contains a proper non-empty regularly closed set is $\mathfrak{M}$-metacompact if and only if every regularly closed subset of X is $\mathfrak{M}$-metacompact.*

PROOF. Since every regularly closed set is closed, therefore the 'only if' part is obvious. We shall now prove the 'if' part. Let U be any proper non-null regularly closed subset of X. Therefore Cl Int $U\subset U$. Let Int $U=V$. Then V is a non-empty open subset of X. Since V is open and $V\cap X-\mathrm{Cl}\,V=\phi$, therefore $V\cap\mathrm{Cl}(X-\mathrm{Cl}\,V)=\phi$. This shows that $\mathrm{Cl}(X-\mathrm{Cl}\,V)$ is a proper regularly closed subset of X. By hypothesis, $\mathrm{Cl}(X-\mathrm{Cl}\,V)$ and $\mathrm{Cl}\,V$ are $\mathfrak{M}$-metacompact. Also $\mathrm{Cl}(X-\mathrm{Cl}\,V)\cup\mathrm{Cl}\,V=X$. Hence X is $\mathfrak{M}$-metacompact by Theorem 3.10.

COROLLARY 3.6. *A weakly regular space X is $\mathfrak{M}$-metacompact if and only if every proper regularly closed subset of X is $\mathfrak{M}$-metacompact.*

COROLLARY 3.7. *A semi-regular space X is $\mathfrak{M}$-metacompact if and only if every proper regularly closed subset of X is $\mathfrak{M}$-metacompact.*

THEOREM 3.12. *Let X be a regular space and let $\mathscr{G}$ be an open basis of neighbourhoods of a point $x\in X$ such that $X-G$ is $\mathfrak{M}$-metacompact for each $G\in\mathscr{G}$, then X is $\mathfrak{M}$-metacompact.*

PROOF. Let $\mathscr{U}=\{U_\alpha:\alpha\in\Lambda\}$ be any open covering of X of cardinality $\leq\mathfrak{M}$. Since X is regular and $x\in U_{\alpha_x}$ for some $\alpha_x\in\Lambda$, therefore there is a $G_x\in\mathscr{G}$ such that $x\in G_x\subset\mathrm{Cl}G_x\subset U_{\alpha_x}$. Since $X-G_x$ is $\mathfrak{M}$-metacompact and $\{(X-G_x)\cap U_\alpha:\alpha\in\Lambda\}$ is an open covering of $X-G_x$ of cardinality $\leq\mathfrak{M}$, there exists a point-finite (in

$X-G_x$) open (in $X-G_x$) refinement $\{V_\beta \cap (X-G_x) : \beta \in \Delta\}$ of $\{(X-G_x) \cap U_\alpha : \alpha \in \Lambda\}$ where each V_β is open in X. Let $V_\beta^* = V_\beta \cap (X - \mathrm{Cl}G_x)$ for each $\beta \in \Delta$. Then $\{V_\beta^* : \beta \in \Delta_x\} \cup \{U_{\alpha_i}\}$ is a point-finite open refinement of $\{U_\alpha : \alpha \in \Lambda\}$ and hence X is $\mathfrak{M}$-metacompact.

COROLLARY 3.8. *If X is a regular space and $\mathscr{G}$ is an open basis of neighbourhoods of a point $x \in X$ such that $X-G$ is metacompact (countably metacompact) for each $G \in \mathscr{G}$, then X is metacompact (countably metacompact).*

Let us call a space *locally $\mathfrak{M}$-metacompact* if every point has a neighbourhood whose closure is $\mathfrak{M}$-metacompact. Mrówka [16] has shown that every completely regular, locally paracompact space can be embedded in a paracompact space as an open subspace. We prove that a similar result also holds for $\mathfrak{M}$-metacompact spaces.

THEOREM 3.13. *Every completely regular locally $\mathfrak{M}$-metacompact space can be embedded in an $\mathfrak{M}$-metacompact space as an open subspace.*

PROOF. We know that every closed subset of an $\mathfrak{M}$-metacompact space is $\mathfrak{M}$-metacompact. Also, by Theorem 3.11, it follows that $\mathfrak{M}$-metacompactness is finitely additive with respect to closed subsets. Again, Theorem 3.12 shows that $\mathfrak{M}$-metacompactness satisfies the embedding condition (W) of Mrówka [16]. Hence the result follows as in [16].

4. $\mathfrak{M}$-metacompact spaces and mappings

THEOREM 4.1. *If f is a closed, continuous mapping of a space X onto an $\mathfrak{M}$-metacompact space Y such that $f^{-1}(y)$ is $\mathfrak{M}$-compact for every point y of Y, then X is $\mathfrak{M}$-metacompact.*

PROOF. Let $\mathscr{U} = \{U_\alpha : \alpha \in \Lambda\}$ be any open covering of X of cardinality $\leq \mathfrak{M}$. Let Δ denote the family of all finite subsets of Λ. Then $|\Delta| \leq \mathfrak{M}$. Since $f^{-1}(y)$ is $\mathfrak{M}$-compact for each $y \in Y$, there is a finite subset γ of Λ such that $f^{-1}(y) \subset \bigcup_{\alpha \in \gamma} U_\alpha$. Let $V_\gamma = Y - f(X - \bigcup_{\alpha \in \gamma} U_\alpha)$. Then V_γ is open and $y \in V_\gamma$ and $f^{-1}(V_\gamma) \subset \bigcup_{\alpha \in \gamma} U_\alpha$. Thus $\mathscr{V} = \{V_\gamma : \gamma \in \Delta\}$ is an open covering of Y of cardinality $\leq \mathfrak{M}$. Since Y is $\mathfrak{M}$-metacompact, there exists a point-finite open refinement $\{W_\delta : \delta \in \Delta'\}$ of $\mathscr{V}$. Since for each δ, there is $\gamma_\delta \in \Delta$ such that $W_\delta \subset V_{\gamma_\delta}$, therefore for each $\delta \in \Delta'$, there exists $\gamma_\delta \in \Delta$ such that $f^{-1}(W_\delta) \subset f^{-1}(V_{\gamma_\delta}) \subset \bigcup_{\alpha \in \gamma_\delta} U_\alpha$. Thus $\{f^{-1}(W_\delta) \cap U_\alpha :$

$\alpha\in\gamma_\delta$, $\delta\in\Delta'\}$ is a point-finite open refinement of $\mathscr{U}$ and hence X is $\mathfrak{M}$-metacompact.

REMARK 4.1. The methods of proofs of Theorems 2.2 and 4.1 are essentially those of Hanai's [8, Theorem 1] and [7, Theorem 1] respectively.

COROLLARY 4.1. [8, Theorem 4]. *If f is a closed, continuous mapping of a space X onto a metacompact space Y such that $f^{-1}(y)$ is compact for each $y\in Y$, then X is metacompact.*

COROLLARY 4.2. *If f is a closed, continuous mapping of a space X onto a countably metacompact space Y such that $f^{-1}(y)$ is countably compact for each $y\in Y$, then Y is countably metacompact.*

COROLLARY 4.3. *Countable metacompactness is a fitting property.*
(For the definition of a fitting property, see Henriksen and Isbell [9]).

COROLLARY 4.4. *If X is an $\mathfrak{M}$-metacompact topological space such that every point of X has a neighbourhood basis with cardinality $\leq\mathfrak{M}$ and Y is an $\mathfrak{M}$-compact space, then the product $X\times Y$ is $\mathfrak{M}$-metacompact.*

PROOF. By Theorem 4 in [7] the projection P of $X\times Y$ onto X is closed. Also, the projections are continuous. For each $x\in X$, $P^{-1}(x)=\{x\}\times Y$. But Y is $\mathfrak{M}$-compact and therefore $\{x\}\times Y$ is $\mathfrak{M}$-compact. Thus P is a closed continuous mapping of $X\times Y$ onto the $\mathfrak{M}$-metacompact space X such that $P^{-1}(x)$ is $\mathfrak{M}$-compact for each $x\in X$. By Theorem 4.1, $X\times Y$ is $\mathfrak{M}$-metacompact.

In particular, we have

COROLLARY 4.5. *If X is a countably metacompact space satisfying the first axiom of countability and if Y is countably compact, then the product space $X\times Y$ is countably metacompact.*

DEFINITION 4.1. [Ponomarev, 17] Let $\mathscr{W}$ be a fixed covering of a space X. A continuous mapping $f: X\to Y$ of the space X onto a space Y is called an *$(\mathscr{W},p)$-mapping* if, for every point $y\in Y$, there exist a subcollection $\mathscr{W}_y$ of $\mathscr{W}$ with the property p in $\cup\{W: W\in\mathscr{W}_y\}$ and a neighbourhood V_y of y in Y such that $f^{-1}(V_y)\subset\cup\{W: W\in\mathscr{W}_y\}$.

Let $\mathscr{W}$ be an open covering of a space X and let f be a continuous mapping of X into a space Y. Then f is said to be a *$\mathscr{W}$-mapping* if there exists an open covering $\mathscr{G}$ of Y such that $\{f^{-1}(G): G\in\mathscr{G}\}$ is a refinement of $\mathscr{W}$. f is said to be a *finite $\mathscr{W}$-mapping* if there is an open covering $\mathscr{G}$ of Y such that each member of the family $\{f^{-1}(G) : G \in \mathscr{G}\}$ is contained in a union of finitely many members of $\mathscr{W}$.

Let $\mathfrak{X}$ be a class of topological spaces containing with any space X all spaces homeomorphic to X. The class of all spaces X such that for any open covering $\mathscr{W}$ of X there exists a $\mathscr{W}$-mapping from Y into a space from $\mathfrak{X}$ is said to be the class $\mathscr{W}$-generated by $\mathfrak{X}$ and is denoted by $0(\mathfrak{X})$. If $0(\mathfrak{X})=\mathfrak{X}$, we say that the class $\mathfrak{X}$ is closed with respect to $\mathscr{W}$-mappings.

THEOREM 4.2. *A sufficient condition for a space X to be metacompact is that for every open covering $\mathscr{W}$ of X, there exists an $(\mathscr{W},p)$-mapping of X onto some metacompact space Y where p is the property of being point-finite.*

PROOF. Let $\mathscr{W}=\{W\}$ be any open covering of X. For each $y \in Y$, let V_y be an open neighbourhood of y such that $f^{-1}(V_y) \subset \bigcup \{W : W \in \mathscr{W}_y\}$ where f is the $(\mathscr{W}, p)$-mapping of X onto the metacompact space Y and $\mathscr{W}_y$ is a point-finite subfamily of $\mathscr{W}$. Then, $\{V_y : y \in Y\}$ is an open covering of Y and therefore there exists a point-finite open refinement $\{U_\alpha : \alpha \in \Lambda\}$ of $\{V_y : y \in Y\}$. For each $\alpha \in \Lambda$, choose y_α such that $U_\alpha \subset V_{y_\alpha}$. Then $f^{-1}(U_\alpha) \subset f^{-1}(V_{y_\alpha}) \subset \bigcup \{W : W \in \mathscr{W}_{y_\alpha}\}$. Then $\mathscr{W}' = \{f^{-1}(U_\alpha) \cap W : W \in \mathscr{W}_{y_\alpha} : \alpha \in \Lambda\}$ is a point-finite open refinement of $\mathscr{W}$ and hence X is metacompact.

THEOREM 4.3. *A sufficient condition for a space X to be $\mathfrak{M}$-metacompact is that for every open covering $\mathscr{W}$ of X of cardinality $\leq \mathfrak{M}$, there exists an $(\mathscr{W},p)$-mapping of X onto some $\mathfrak{M}$-metacompact space Y, where p is the property of being finite.*

PROOF. Let $\mathscr{W}=\{W_\alpha : \alpha \in \Lambda\}$ be any open covering of X with $|\Lambda| \leq \mathfrak{M}$. Let Δ be the family of all finite subsets of Λ. Then $|\Delta| \leq \mathfrak{M}$. For each $y \in Y$, let U_y be an open neighbourhood of y such that $f^{-1}(U_y) \subset \bigcup \{W_\alpha : \alpha \in \gamma\}$ for some $\alpha \in \Lambda$. Let $\mathscr{U}=\{U\}$ be the open covering of Y by such open sets. Let the index set Δ be well ordered and for each U, let Δ_U be the first γ such that $f^{-1}(U) \subset \bigcup \{W_\alpha : \alpha \in \gamma\}$. For any γ, let V_γ be the union of all those U's for which $f^{-1}(U) \subset \bigcup \{W_\alpha : \alpha \in \gamma\}$. Then $f^{-1}(V_\gamma) \subset \{W_\alpha : \alpha \in \gamma\}$ and $\{V_\gamma : \gamma \in \Delta\}$ is an open covering of Y of cardinality $\leq \mathfrak{M}$. Let $\{G_\gamma : \gamma \in \Delta\}$ be a point-finite open refinement of $\{V_\gamma : \gamma \in \Delta\}$. Then $\mathscr{W}'=\{f^{-1}(G_\gamma) \cap W_\alpha : \alpha \in \gamma,\ \gamma \in \Delta\}$ is a point-finite open refinement of $\mathscr{W}$ Hence X is $\mathfrak{M}$-metacompact.

COROLLARY 4.6. *The class of $\mathfrak{M}$-metacompact (metacompact, countably metacompact) spaces is closed with respect to $\mathscr{W}$-mappings.*

COROLLARY 4.7. *The class of $\mathfrak{M}$-metacompact (metacompact, countably metacompact) spaces is closed with respect to finite $\mathscr{W}$-mapping.*

5. $\mathfrak{M}$-metacompactness and subsets.

THEOREM 5.1. *If every open subspace of a space X is $\mathfrak{M}$-metacompact, then every subspace of X is $\mathfrak{M}$-metacompact.*

PROOF. Let A be any subspace of an $\mathfrak{M}$-metacompact space X. Let $\mathscr{U}$ be a relatively open cover of A of cardinality $\leq\mathfrak{M}$. Let $\mathscr{U}^*$ be a collection of open subsets of X such that $\mathscr{U}=\{U^*\cap A : U^*\in\mathscr{U}^*\}$. Then $G=\bigcup\{U^* : U^*\in\mathscr{U}\}$ is an open subspace and hence G is $\mathfrak{M}$-metacompact. Thus the open covering $\{U^* : U^*\in\mathscr{U}^*\}$ of G of cardinality $\leq\mathfrak{M}$ has a point-finite open refinement $\mathscr{H}^*$ of $\mathscr{U}^*$. Let $\mathscr{H}=\{H^*\cap A : H^*\in\mathscr{H}^*\}$. Then $\mathscr{H}$ is a point-finite open refinement of $\mathscr{U}$ and hence A is $\mathfrak{M}$-metacompact.

COROLLARY 5.1. *A space X is hereditarily countably metacompact (metacompact) if every open subspace of it is countably metacompact (metacompact).*

A space X is said to be *totally normal* if every open subset G of X is expressible as a locally finite (in G) union of open F_σ-subsets of X.

COROLLARY 5.2. *A totally normal countably metacompact space is hereditarily countably metacompact.*

PROOF. In view of Corollary 5.1 we only have to prove that every open subset of a totally normal countably metacompact space X is countably metacompact. Let G be an open subset of X. Then $G=\bigcup\{V_\alpha : \alpha\in\Lambda\}$ where $\{V_\alpha : \alpha\in\Lambda\}$ is a locally finite (in G) family of open F_σ-subsets of X. For each $\alpha\in\Lambda$ let $\{F_{\alpha,i}\}$ be a sequence of closed subsets of X such that $V_\alpha=\bigcup_{i=1}^{\infty}F_{\alpha,i}$. Since X is normal, for each i there exists an open subset $W_{\alpha,i}$ such that $F_{\alpha,i}\subset W_{\alpha,i}\subset \mathrm{Cl}\, W_{\alpha,i}\subset V_\alpha$. For each i, let $\mathscr{W}_i=\{W_{\alpha,i} : \alpha\in\Lambda\}$. Then $\mathscr{W}=\bigcup_{i=1}^{\infty}\mathscr{W}_i$ is a σ-locally finite (in G) open cover of G such that the closure of each member of $\mathscr{W}$, being a closed subset of a countably metacompact space, is countably metacompact. Hence G is countably metacompact, in view of Corollary 2.1. Hence the result.

Corollary 5.2 is proved for metacompact spaces by Hodel **in** [10].

THEOREM 5.2. *A subset A of a space X is $\mathfrak{M}$-metacompact if and only if for each open set G containing A there is an $\mathfrak{M}$-metacompact subset Y such that $A\subset Y\subset G$.*

PROOF. Only the 'if' part need be proved. Let $\{U_\alpha\cap A : \alpha\in\Lambda\}$ be a relatively open covering of A with $|\Lambda|\leq\mathfrak{M}$. Let $G=\bigcup_{\alpha\in\Lambda} U_\alpha$. Then G is an open set containing A and therefore there exists an $\mathfrak{M}$-metacompact subset Y such that $A\subset Y\subset G$. Then, $\{U_\alpha\cap Y : \alpha\in\Lambda\}$ is an open cover of Y of cardinality $\leq\mathfrak{M}$. Let $\{V_\beta\cap Y : \beta\in\Delta\}$ be a point-finite (in Y) open (in Y) refinement of $\{U_\alpha\cap Y : \alpha\in\Lambda\}$ such that each V_β is open in X. Then $\{A\cap V_\beta : \beta\in\Delta\}$ is a point-finite open refinement of $\{U_\alpha\cap A : \alpha\in\Lambda\}$ and hence A is $\mathfrak{M}$-metacompact.

COROLLARY 5.3. *A subset A of a space X is metacompact (countably metacompact) if and only if every open set containing A contains a metacompact (countably metacompact) set containing A.*

A subset A of a space X is called a *generalized F_σ-subset* if for every open set U containing A there is an F_σ-subset B such that $A\subset B\subset U$.

THEOREM 5.3. *Every generalized F_σ-subset of a countably metacompact space is countably metacompact.*

PROOF. Let $\mathscr{U}=\{U_n : n\in N\}$ be a countable open covering of a generalized F_σ-subset of Y of a countably metacompact space X. Then for each $n\in N$, $U_n=V_n\cap Y$, where V_n is an open subset of X. Let $G=\bigcup_{n\in N} V_n$. Since G is an open subset of X containing Y and Y is a generalized F_σ-subset, there exists a sequence $\{F_n\}$ of closed subsets of X such that $Y\subset \bigcup_{n\in N} F_n\subset G$. For each $n\in N$, $\{V_n : n\in N\}\cup\{X-F_n\}$ is then a countable open covering of X. Hence, for each $n\in N$ there is a point-finite open refinement $\{W_{n,m} : m\in N\}\cup\{X-F_n\}$ such that $W_{n,m}\subset V_m$ for each m. Let $\mathscr{W}=\{W_{n,m}-\bigcup_{k\leq n} F_k : n, m\in N\}$. Then $\mathscr{W}$ is point-finite, since $\{Y-\bigcup_{k\leq n} F_k : n\in N\}$ and $\{W_{n,m} : m\in N\}$ are point-finite. Also $\mathscr{W}$ is an open refinement of $\mathscr{U}$ and hence Y is countably metacompact.

COROLLARY 5.4. *Every generalized F_σ-subset of a metacompact space is metacompact.*

THEOREM 5.4. *Every generalized F_σ-subset of an $\mathfrak{M}$-metacompact space is $\mathfrak{M}$-metacompact.*

PROOF. Let $\{U_\alpha : \alpha\in\Lambda\}$ be any open covering of a generalized F_σ-subset Y of an $\mathfrak{M}$-metacompact space X. Let $|\Lambda|\leq\mathfrak{M}$. For each $\alpha\in\Lambda$, there is an open subset G_α of X such that $U_\alpha=G_\alpha\cap Y$. Set $G=\bigcup_{\alpha\in\Lambda}G_\alpha$. Let $\{F_n : n\in N\}$ be a sequence of closed subsets of X such that $Y\subset\bigcup_{n\in N}F_n\subset G$. Then, by $\mathfrak{M}$-metacompactness of X, there exists for each $n\in N$, a point-finite open covering $\{V_{n,\alpha}\cap Y : \alpha\in\Lambda\}\cup\{X-F_n\}$ of X such that $V_{n,\alpha}\subset G_\alpha$ for all $\alpha\in\Lambda$. Then $\{V_{n,\alpha}\cap Y : n\in N,\ \alpha\in\Lambda\}$ is a σ-point-finite open (in Y) refinement of $\{U_\alpha : \alpha\in\Lambda\}$. Also, since X is in particular countably metacompact, therefore, by Theorem 5.3, Y is countably metacompact. Hence by Theorem 3.1, Y is $\mathfrak{M}$-metacompact.

COROLLARY 5.5. *Every generalized cozero-subspace of an $\mathfrak{M}$-metacompact (metacompact) space is $\mathfrak{M}$-metacompact (metacompact).*

PROOF. Trivial, since every cozero subset is an F_σ-subset.

COROLLARY 5.6. *Every subset of a perfectly normal $\mathfrak{M}$-metacompact (metacompact) space is $\mathfrak{M}$-metacompact (metacompact).*

PROOF. Every open subspace of a perfectly normal space is a cozero set and hence a generalized cozero set. Thus, by Corollary 5.5, every open subspace of a perfectly normal $\mathfrak{M}$-metacompact (metacompact) space is $\mathfrak{M}$-metacompact (metacompact). The required result then follows from Theorem 5.1.

REMARK 5.1. For a countably metacompact space, Theorem 5.4 follows as a corollary to Corollary 5.2, since every perfectly normal space is totally normal.

THEOREM 5.5. *Let X be a perfectly normal space. If $S=\bigcup_{i\in N}S_i$ be the union of countably many open subsets S_i of X, then X is $\mathfrak{M}$-metacompact if and only if each S_i is $\mathfrak{M}$-metacompact.*

PROOF. The 'only if' part follows from Corollary 5.6. We shall prove the 'if' part. Since S is perfectly normal, the countable open covering $\{S_i\}$ of S has a locally finite refinement $\{T_i\}$ such that $T_i\subset S_i$, since S is perfectly normal. Let $\{U_\alpha : \alpha\in\Lambda\}$ be any open covering of S by open subsets of S of cardinality $\leq\mathfrak{M}$. Then for each i, $\{S_i\cap U_\alpha : \alpha\in\Lambda\}$ is a relatively open covering of S_i of cardinality $\leq\mathfrak{M}$, and since each S_i is $\mathfrak{M}$-metacompact there exists a point-finite (in S_i) family $\mathscr{V}_i=\{V_{i,\beta} : \beta\in\Delta_i\}$ of open subsets of S_i which cover S_i and which refines $\{S_i\cap U_\alpha : \alpha\in\Lambda\}$. Then for each i, $\{T_i\cap V_{i,\beta} : \beta\in\Delta_i\}$ is an open covering of T_i. Also, this is a locally finite and hence also a point-finite open refinement of $\{T_i\cap U_\alpha : \alpha\in\Lambda\}$. Let $\mathscr{V}=\{T_i\cap V_{i,\beta} : \beta\in\Delta_i,\ \alpha\in\Lambda\}$. Then $\mathscr{V}$ is a point-finite open refinement of $\{U_\alpha : \alpha \in \Lambda\}$. Hence S is $\mathfrak{M}$-metacompact.

COROLLARY 5.7. *Let X be a perfectly normal space. If $X = \bigcup_{i \in N} S_i$ where S_i are open subsets of X, then X is $\mathfrak{M}$-metacompact (metacompact) if and only if each S_i is $\mathfrak{M}$-metacompact (metacompact).*

6. Simple extensions and $\mathfrak{M}$-metacompactness.

If $(X, \mathscr{T})$ is any topological space and A is a subset of X such that $A \notin \mathscr{T}$ then the topology $\mathscr{T}(A) = \{U \cup (V \cap A) : U, V \in \mathscr{T}\}$ is called a *simple extension* of $\mathscr{T}$. The concept of simple extensions is due to Levine [12]. Simple extensions have been studied in greater details by Borges in [3] where necessary and sufficient conditions have been given for $(X, \mathscr{T}(A))$ to inherit certain topological properties from $(X, \mathscr{T})$. In the present section we obtain necessary and sufficient conditions for $(X, \mathscr{T}(A))$ to be $\mathfrak{M}$-metacompact if $(X, \mathscr{T})$ is $\mathfrak{M}$-metacompact.

THEOREM 6.1. *Let $(X, \mathscr{T})$ be any space and let $\mathscr{T}(A)$ be a simple extension of $\mathscr{T}$. If $X - A \in \mathscr{T}$, then $(X, \mathscr{T}(A))$ is $\mathfrak{M}$-metacompact if and only if $(X - A, \mathscr{T} \cap (X - A))$ is $\mathfrak{M}$-metacompact.*

PROOF. Let $(X, \mathscr{T}(A))$ be $\mathfrak{M}$-metacompact. Since $X - A$ is a closed subspace of $(X, \mathscr{T}(A))$, it follows that $(X - A, \mathscr{T}(A) \cap (X - A))$ is $\mathfrak{M}$-metacompact. But $(X - A, \mathscr{T} \cap (X - A)) = (X - A, \mathscr{T}(A) \cap (X - A))$, therefore $(X - A, \mathscr{T} \cap (X - A))$ is $\mathfrak{M}$-metacompact. Conversely, let $(X - A, \mathscr{T} \cap (X - A))$ be $\mathfrak{M}$-metacompact. $(A, \mathscr{T} \cap A)$, being a closed subspace of $(X, \mathscr{T})$ is $\mathfrak{M}$-metacompact. Hence X is the union of two disjoint $\mathfrak{M}$-metacompact $\mathscr{T}(A)$-open subspaces A and $X - A$. Thus $(X, \mathscr{T}(A))$ is $\mathfrak{M}$-metacompact by Corollary 3.4.

COROLLARY 6.1. *Let $\mathscr{T}(A)$ be a simple extension of a topology $\mathscr{T}$ on X and let $X - A \in \mathscr{T}$. Then $(X, \mathscr{T}(A))$ is metacompact if and only if $(X - A, \mathscr{T} \cap (X - A))$ is metacompact.*

THEOREM 6.2. *Let $(X, \mathscr{T})$ be an $\mathfrak{M}$-metacompact regular space and A be a subset of X. Then $(X, \mathscr{T}(A))$ is an $\mathfrak{M}$-metacompact regular space if and only if it is a regular space and $X - A$ is an $\mathfrak{M}$-metacompact subspace of $(X, \mathscr{T})$.*

PROOF. The "only if" part is obvious, since $X - A$ is a closed subspace of $(X, \mathscr{T}(A))$. To prove the "if" part, let $\mathscr{U}$ be a $\mathscr{T}(A)$-open covering of X of car dinality $\leq \mathfrak{M}$. Without any loss of generality we assume that for each $x \in X - A$ there exists some $U \in \mathscr{U} \cap \mathscr{T} = \{G \cap H : G \in \mathscr{U}, H \in \mathscr{T}\}$ such that $x \in U$ and $U \cap A$

$=\phi$, for each $a\in A$ there exists some $U\in\mathcal{T}$ such that $U\cap A\in\mathcal{U}$, and for each $y\in \mathrm{Cl}A-A$ there exists some $V\in\mathcal{T}\cap\mathcal{U}$ such that $y\in V$. Let $\mathcal{V}=\{U\in\mathcal{U} : U\in\mathcal{T}$ and $U\cap(\mathrm{Cl}A-A)\neq\phi\}$. By Theorem 3.2 in [3], $\mathrm{Cl}A-A$ is $\mathcal{T}$-closed and hence $\mathcal{V}\cup(X-(\mathrm{Cl}A-A))$ is a $\mathcal{T}$-open cover of cardinality $\leq\mathfrak{M}$ of the $\mathfrak{M}$-metacompact space $(X,\mathcal{T})$. Let $\mathcal{V}'$ be a point-finite $\mathcal{T}$-open refinement of $\mathcal{V}\cup(X-(\mathrm{Cl}A-A))$. Let $\mathcal{W}'=\{W\in\mathcal{V}' : W\cap(\mathrm{Fr}A-A)\neq\phi\}$. Then $\mathcal{W}'$ is a point finite family of $\mathcal{T}$-open subsets of X which covers $\mathrm{Fr}A-A$ and refines $\mathcal{V}$. Let $W'=\bigcup\mathcal{W}'$. Then $A-W'$ is a $\mathcal{T}$-closed subset of X, since $A-W'=(A\cup\mathrm{Fr}A)-W'=\mathrm{Cl}A-W'$ (as $W'\supset\mathrm{Fr}A-A$). Let $\mathcal{G}=\{V\in\mathcal{T} : V\cap A\in\mathcal{U}$ and $V\cap(A-W')\neq\phi\}$. Then as above we can find a point-finite family $\mathcal{G}'$ of $\mathcal{T}$-open subsets of X which covers $A-W'$ and refines $\mathcal{G}$. Hence $\mathcal{G}''=\{V\cap A : V\in\mathcal{G}'\}$ is a point-finite family of $\mathcal{T}(A)$-open subsets of X which refines $\{V\cap A\in\mathcal{U} : V\cap(A-\bigcup\mathcal{V})\neq\phi\}$. Again, let $\mathcal{H}=\{U\in\mathcal{U} : U\in\mathcal{T},\ U\cap(X-(W'\cup A))\neq\phi$ and $U\cap A=\phi\}$. Then $\mathcal{H}$ covers $X-(W'\cup A)$ since $W'\cup A\supset\mathrm{Cl}A\cup W'$. Also the cardinality of $\mathcal{H}$ is $\leq\mathfrak{M}$. Since $X-(W'\cup A)$ is a closed subset of $(X-A,\ \mathcal{T}\cap(X-A))$ there exists a point-finite family $\mathcal{H}'$ of $\mathcal{T}$-open subsets of X which covers $X-(W'\cup A)$ and refines $\mathcal{H}$. Now $\mathcal{U}'=\mathcal{W}'\cup\mathcal{G}''\cup\mathcal{H}'$ is a point-finite $\mathcal{T}(A)$-open refinement of $\mathcal{U}$.

COROLLARY 6.2. *If $(X,\mathcal{T})$ is a metacompact regular space and A is a subset of X, then $(X,\mathcal{T}(A))$ is a metacompact regular space if and only if it is a regular space and $X-A$ is a metacompact subspace of $(X,\mathcal{T})$.*

THEOREM 6.3. *Let $(X,\mathcal{T})$ be hereditarily $\mathfrak{M}$-metacompact regular space and A a subset of X. Then $(X,\mathcal{T}(A))$ is hereditarily $\mathfrak{M}$-metacompact and regular.*

PROOF. Let Y be any subset of X. Then $(Y,(\mathcal{T}\cap Y)(A\cap Y))=(Y,\mathcal{T}(A)\cap Y)$. Therefore $(Y,\mathcal{T}(A)\cap Y)$ is $\mathfrak{M}$-metacompact follows from the above Theorem 6.2.

COROLLARY 6.3. *Let $(X,\mathcal{T})$ be a hereditarily metacompact regular space and let A be a subset of X. Then $(X,\mathcal{T}(A))$ is a hereditarily metacompact regular space.*

THEOREM 6.4. *If $(X,\mathcal{T})$ is hereditarily $\mathfrak{M}$-metacompact and A is a subset of X such that $X-A\in\mathcal{T}$ then $(X,\mathcal{T}(A))$ is hereditarily $\mathfrak{M}$-metacompact.*

PROOF. Since $(X,\mathcal{T})$ is hereditarily $\mathfrak{M}$-metacompact, therefore $(X-A,\mathcal{T}\cap(X-A))$ and $(A,\mathcal{T}\cap A)$ are also hereditarily $\mathfrak{M}$-metacompact. But $(A,\mathcal{T}\cap A)=(A,\mathcal{T}(A)\cap A)$ and $(X-A,\ \mathcal{T}\cap(X-A))=(X-A,\ \mathcal{T}(A)\cap(X-A))$. Thus X is the union of two disjoint $\mathcal{T}(A)$-open hereditarily $\mathfrak{M}$-metacompact subspaces A and $X-A$. Hence $(A,\mathcal{T}(A))$ is hereditarily $\mathfrak{M}$-metacompact.

COROLLAR 6.4. *If $(X, \mathscr{T})$ is hereditarily metacompact and A is a subset of X such that $X-A\in\mathscr{T}$, then $(X, \mathscr{T}(A))$ is hereditarily metacompact.*

7. Invertibility and $\mathfrak{M}$-metacompactness.

DEFINITION 7.1. [Doyle and Hocking, 4]. A space X is said to be *invertible* if for each non-empty open subset U of X, there exists a homeomorphism h of X such that $h(X-U)\subset U$. h is called an *inverting homeomorphism* for U.

DEFINITION 7.2. [Hong, 11]. A space X is said to be a *generalized invertible space* if there exists a proper open subset U of X and there is a homeomorphism h of X onto X such that for each $x\in X$, $h^n(x)\subset U$ for some integer n. The pair (U, h) is called an *inverting pair* for X.

THEOREM 7.1. *If $(X, \mathscr{T})$ is a countably metacompact generalized invertible space and (U, h) is an inverting pair for X and $U\subset A$ where A is $\mathfrak{M}$-metacompact, then X is $\mathfrak{M}$-metacompact.*

PROOF. Let $\mathscr{U}=\{U_\alpha : \alpha\in\Lambda\}$ be any open covering of X of cardinality $\leq\mathfrak{M}$. For each integer n, $\{h^n(U_\alpha)\cap A : \alpha\in\Lambda\}$ is a relatively open covering of A of cardinality $\leq\mathfrak{M}$. Since A is $\mathfrak{M}$-metacompact, there exists a point-finite family $\{V_\beta\cap A : \beta\in\Delta^n\}$ which refines $\{h^n(U_\alpha)\cap A : \alpha\in\Lambda\}$ and each V_β is an open subset of X. Then for each integer n, $\{V_\beta\cap U : \beta\in\Delta^n\}$ is a point-finite family of open subsets of X. Consider the family $\mathscr{V}^n=\{h^n(V_\beta\cap U) : \beta\in\Delta^n\}$ which is a point-finite family of open subsets of X. Also, for each $\beta\in\Delta^n$, there is an $\alpha\in\Lambda$ such that $V_\beta\cap U\subset h_n(U_\alpha)\cap A$. Therefore $h^{-n}(V_\beta\cap U)\subset h^{-n}(h^n(U_\alpha)\cap A)\subset U_\alpha$. Thus $\mathscr{V}=\bigcup_{n=1}^{\infty}\mathscr{V}^n$ is a σ-point-finite open refinement of $\mathscr{U}$ and hence X is $\mathfrak{M}$-metacompact by Theorem 3.1.

THEOREM 7.2. *Let X be an invertible space and let U be an open $\mathfrak{M}$-metacompact subspace. Then X is $\mathfrak{M}$-metacompact.*

PROOF. Let h be an inverting homeomorphism of U. Then $f(U)$ is open and $X=U\cup f(U)$. Since U and $f(U)$ are both $\mathfrak{M}$-metacompact and open, the result follows from Corollary 3.4.

COROLLARY 7.1. *Let X be an invertible space and let U be an open metacompact subspace. Then X is metacompact.*

9. Almost $\mathfrak{M}$-metacompact spaces.

A space X is called *almost metacompact* if every open covering of X admits of a point-finite weak refinement whose closures cover the space X.

Almost $\mathfrak{M}$-metacompact spaces are, then, defined analogously.

Below we give an example to show the existence of an almost metacompact space which is not metacompact.

EXAMPLE 8.1. An almost metacompact space which is not metacompact. Let X be an uncountable set and let $p \in X$. Let $\mathscr{T}$ consist of all supersets of p and empty set. Then $(X, \mathscr{T})$ is not metacompact, since the open cover $\{\{p, x\} : x \in X\}$ admits of no point-finite open refinement. But, since $\mathrm{Cl}\{p\} = X$, the space is almost metacompact.

THEOREM 8.1. *Let $\{G_\alpha : \alpha \in \Lambda\}$ be a point-finite open covering of X. If each $\mathrm{Cl}\, G_\alpha$ is almost $\mathfrak{M}$-metacompact, then X is almost $\mathfrak{M}$-metacompact.*

PROOF. Let $\{U_\beta : \beta \in \Delta\}$ be any open covering of X of cardinality $\leq \mathfrak{M}$. Then, for each $\alpha \in \Lambda$, $\{U_\beta \cap \mathrm{Cl}\, G_\alpha : \beta \in \Delta\}$ is a relatively open covering of $\mathrm{Cl}\, G_\alpha$ of cardinality $\leq \mathfrak{M}$. Since $\mathrm{Cl}\, G_\alpha$ is $\mathfrak{M}$-metacompact, there exists a point-finite open (in $\mathrm{Cl}\, G_\alpha$) weak refinement $\{V_\gamma \cap \mathrm{Cl}\, G_\alpha : \gamma \in \Gamma_\alpha\}$ of $\{U_\beta \cap \mathrm{Cl}\, G_\alpha : \beta \in \Delta\}$ such that each V_γ is open in X and the closures of $V_\gamma \cap \mathrm{Cl}\, G_\alpha$ in $\mathrm{Cl}\, G_\alpha$ cover $\mathrm{Cl}\, G_\alpha$. Then the family $\{V_\gamma \cap G_\alpha : \gamma \in \Gamma_\alpha ; \alpha \in \Lambda\}$ is a family of open subsets of X which is point-finite and whose closures cover X, since $\{G_\alpha : \alpha \in \Lambda\}$ is a covering of X and $\mathrm{Cl}\,(V_\gamma \cap \mathrm{Cl}\, G_\alpha) = \mathrm{Cl}\,(V_\gamma \cap G_\alpha)$, V_γ being open. Hence X is $\mathfrak{M}$-metacompact.

THEOREM 8.2. *If $S \subset X$ where $S = \bigcup_{\alpha \in \Lambda} S_\alpha$ and each S_α is open in S and $\{S_\alpha : \alpha \in \Lambda\}$ is point-finite (in S), then S is almost $\mathfrak{M}$-metacompact if each S_α is almost $\mathfrak{M}$-metacompact.*

PROOF. Let $\{S \cap U_\beta : \beta \in \Delta\}$ be any relatively open covering of S with $|\Delta| \leq \mathfrak{M}$. Then, for each $\alpha \in \Lambda$, $\{S_\alpha \cap U_\beta : \beta \in \Delta\}$ is a relatively open covering of S_α of cardinality $\leq \mathfrak{M}$. Since S_α is almost $\mathfrak{M}$-metacompact, there exists a point-finite (in S_α) family $\mathscr{V}_\alpha = \{V_{\alpha\gamma} : \gamma \in \Delta_\alpha\}$ of open subsets of S_α which weakly refines $\{S_\alpha \cap U_\beta : \beta \in \Delta\}$, hence also $\{U_\beta \cap S : \beta \in \Delta\}$ and whose closures in S_α(and hence in S) cover S_α. Then, $\mathscr{V} = \{V_{\alpha\gamma} : \alpha \in \Lambda,\ \gamma \in \Delta_\alpha\}$ is a point-finite family of open subsets of S which weakly refines $\{S \cap U_\beta : \beta \in \Delta\}$ whose closures in S cover S. Hence S is almost $\mathfrak{M}$-metacompact.

COROLLARY 8.1. *If each member of a point-finite open covering of a space is almost $\mathfrak{M}$-metacompact, then the space is almost $\mathfrak{M}$-metacompact.*

REMARK 8.1. The results of Theorem 8.2 and Corollary 8.1 remain true if almost $\mathfrak{M}$-metacompact be replaced by almost metacompact.

THEOREM 8.3. *If X is almost metacompact, Y is almost compact, then $X \times Y$ is almost metacompact.*

PROOF. Let $\mathscr{U}$ be any open covering of $X \times Y$. For each $z=(x,y)$ in $X \times Y$, there are open sets V_z and W_z in X and Y respectively such that $(x,y) \in V_z \times W_z \subset U$ for some $U \in \mathscr{U}$. For each $x \in X$, denote the set $\{x\} \times Y$ by E_x. The family $\{W_z : z \in E_x\}$ will then form an open cover of the almost compact space Y. Therefore there is a finite subset F_x of E_x such that $\{\mathrm{Cl}\ W_z : z \in F_x\}$ covers Y. Let $V_x = \bigcap \{V_z : z \in F_x\}$. Then $\mathscr{V} = \{V_x : x \in X\}$ is an open covering of the almost-metacompact space X. Thus there is a point-finite family $\mathscr{V}^*$ of open subsets of X which weakly refines $\mathscr{V}$ and the family $\{\mathrm{Cl}\ V : V \in \mathscr{V}^*\}$ covers X. Since $\mathscr{V}^*$ weakly refines $\mathscr{V}$, therefore for each $V \in \mathscr{V}^*$ there exists an $x_v \in X$ such that $V \subset V_{x_v}$. Then, $\mathscr{G} = \{V \times W_z : V \in \mathscr{V}^*, z \in F_x\}$ is a point-finite family of open subsets of $X \times Y$ which weakly refines $\mathscr{U}$ and whose closures cover $X \times Y$. Hence $X \times Y$ is almost metacompact.

COROLLARY 8.2. *The product of an almost metacompact and a compact space is almost metacompact.*

COROLLARY 8.3. *The product of a metacompact and an almost compact space is almost metacompact.*

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REFERENCES

[1] P.R. Andenaes, *Note on metrization*, Fund. Math. 92(1968), 287—292.

[2] R. Arens and J. Dugundji, *Remark on the concept of compactness*, Portugalia Math. 9 (1950), 141—143.

[3] C.J.R. Borges, *On extensions of topologies*, Canad. J. Math. 19(1967), 474—487.

[4] P.H. Doyle and J.G. Hocking, *Invertible spaces*, Amer. Math. Monthly 68(1961), 959—965.

[5] J. Greever, *On screenable topological spaces*, Proc. Japan Acad. 44(1968), 434—438.

[6] Y. Hayashi, *On countably metacompact spaces*, Bull. of Univ. of Osaka prefecture 8(1960), 161—164.

[7] S. Hanai, *Inverse images of closed mappings I*, Proc. Japan Acad. 37(1961), 298—301.

[8] S. Hanai, *On closed mapping, II*, Proc. Japan Acad. 32(1956), 388—391.

[9] M. Henriksen, and J.R. Isbell, *Some Properties of compactifications*, Duke Math. Journal 25(1958), 83—106.

[10] R.E. Hodel, *Sum theorems for topological spaces*, Pacific J. Math. 30(1969), 59—65.

[11] D.X. Hong, *Generalized invertible spaces*, Amer. Math. Monthly 73(1966), 150—154.

[12] N. Levine, *Simple extensions of topologies*, Amer. Math. Monthly 71(1964), 22—25.

[13] E.A. Michael, *Point-finite and locally finite coverings*, Canad. J. Math. 7(1955), 275—279.

[14] L.L. Krajewski, *On expanding locally finite coverings*, Canad. J. Math. 23(1971), 58—68.

[15] K. Morita, *Paracompactness and product spaces*, Fund. Math. 50(1961), 223—236.

[16] S. Mrowka, *On local topological properties*, Bull. Acad. Polon. Sci. 5(1957), 951—956.

[17] V.I. Ponomarev, *On (w, p)-mappings of topological spaces*, (Russian) Dokl. Akad. Nauk SSSR 162(1965), 1251—1255 Soviet Math. Dokl. 6(1965), 858—861.

[18] M.K. Singal and Shashi Prabha Arya, *On $\mathfrak{M}$-paracompact spaces*, Math. Ann. 181 (1969), 119—133.

[19] M.K. Singal and Shashi Prabha Arya, *Two sum theorems for topological spaces*, Israel J. Math. 8(1970), 155—158;

[20] M.K. Singal and Shashi Prabha Arya, *More sum theorems for topological spaces*, Unpublished.

[21] M.K. Singal and Shashi Prabha Arya, *A note on the locally finite sum theorem*, Mathematika. To appear.

[22] M.K. Singal and Pushpa Jain, *On $\mathfrak{M}$-subparacompact spaces*. To appear.