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ON STRONGLY CONTINUOUS MAPPINGS

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The concept of strongly continuous mappings was introduced by Levine [3]. These mappings were also considered by Cullen [2] and Naimpally [6]. Naimpally generalized some results on the function space of continuous mappings to the function space of strongly continuous mappings. It will be seen that even a homeomorphism may fail to be strongly continuous and consequently, many of the mappings that we come across do not possess strong continuity. However, the range of a strongly continuous mapping is determined by the component of its domain and from this point of view, the study of strongly continuous mappings seems to be interesting. We shall divide the paper into six sections. Some characterizations and related results will be obtained in section 1. In section 2, the algebra of strongly continuous mappings will be investigated. Section 3 will be concerned with the study of the behaviour of strong continuity in relation to connectedness. In section 4, strong continuity in relation to compact mappings will be studied, and in section 5 some preservation results will be obtained.
In the last section, the concept of completely continuous mappings will be introduced and studied. The class of completely continuous mappings contains properly the class of strongly continuous mappings.

1. Definition and characterizations of strong continuity

DEFINITION 1.1. [Levine, 3] A mapping $f: X \rightarrow Y$ is said to be *strongly continuous* if for every subset A of X, $f(A^{-}) \subset f(A)$.

Levine proved that $f: X \rightarrow Y$ is strongly continuous if and only if the inverse image of every subset of Y is open (or closed). It follows therefore that f is strongly continuous if and only if the inverse image of every set is open as well as closed.

Obviously, every strongly continuous mapping is continuous. A continuous mapping however, may fail to be strongly continuous as is shown by the following examples:

EXAMPLE 1.1. Let R be the set of reals with co-countable topology and let

$Y=\{a, b, c\}$ with topology $\mathscr{T}^*=\{Y, \phi, \{c\}\}$. Then the mapping f defined by

$$f(x)=\begin{cases} a & \text{if } x \text{ is rational} \\ b & \text{if } x \text{ is irrational} \end{cases}$$

is continuous but not strongly continuous.

EXAMPLE 1.2. Let $X=\{a, b, c\}$ and let $\mathscr{T}$ be the indiscrete topology for X. Then the identity mapping of $(X, \mathscr{T})$ onto $(X, \mathscr{T})$ is a continuous mapping which is not strongly continuous.

Example 1.2 proves also that even a homeomorphism may fail to be strongly continuous.

The following theorem is immediate in view of the definition.

THEOREM 1.1. *A mapping $f: X\to Y$ is strongly continuous if and only if $f^{-1}(y)$ is open for each $y\in Y$.*

COROLLARY 1.1. *A mapping $f: X \xrightarrow{onto} Y$ is strongly continuous if and only if each inverse set is open as well as closed.*

COROLLARY 1.2. *Corresponding to each decomposition of a space X into disjoint open sets, there exists a strongly continuous mapping on X.*

COROLLARY 1.3. *The range of every strongly continuous mapping on a lightly compact space (that is, a space in which every locally finite family of open sets is finite) is finite.*

PROOF. No infinite family of disjoint open sets in a lightly compact space is locally finite. Therefore a pseudo-compact space has decompositions into open sets and these decompositions have finite number of members. The result now follows easily in view of corollary 1.3.

It should be noted that a mapping $f: X\to Y$ such that $f^{-1}(y)$ is closed for each $y\in Y$ need not necessarily be strongly continuous as is shown by the following example.

EXAMPLE 1.3. Let X be the set of real numbers and let be the usual topology for X. If $i: (X,\mathscr{U})\to(X,\mathscr{U})$ be the identity mapping, then it is such that $i^{-1}(y)$ is closed for each $y\in X$. However, i is not strongly continuous.

THEOREM 1.2. *A mapping $f: X\to Y$ is strongly continuous if and only if the decomposition space generated by f is a discrete space.*

PROOF. If f is strongly continuous, then each member of the decomposition is open and closed. Consequently each point in the decomposition space is open and closed and hence the decomposition space is discrete. The 'if' part is obvious.

DEFINITION 1.2. [Levine, 4] A mapping f from X to Y is said to be *weakly continuous* if for each point $x\in X$ and each open set H containing $f(x)$, there is an open set G containing x such that $f(G)\subset\overline{H}$.

THEOREM 1.3. *Every weakly continuous mapping into a discrete space is strongly continuous.*

PROOF. It is easy to prove.

2. Algebra of strongly continuous mappings

THEOREM 2.1. *Restriction of a strongly continuous mapping $f: X\to Y$ to any subset of X is strongly continuous.*

PROOF. Let A be any subset of X. For any point $y\in Y$, $(f|A)^{-1}(y)=f^{-1}(y)\cap A$. Since f is strongly continuous, therefore $f^{-1}(y)$ is an open subset of X. It follows that $f^{-1}(y)\cap A$ is a relatively open subset of A. Hence $f|A$ is strongly continuous.

THEOREM 2.2. *If $f: X\to Y$ is a strongly continuous mapping and $g: Y\to Z$ is any mapping, then $g\circ f: X\to Z$ is strongly continuous.*

PROOF. Let A be any subset of Z. Then $g^{-1}(A)$ is a subset of Y. Since f is strongly continuous, therefore $f^{-1}(g^{-1}(A))$ is an open subset of X, that is, $(g\circ f)^{-1}(A)$ is an open subset of X. Hence $g\circ f$ is strongly continuous.

COROLLARY 2.1. *The composite of two strongly continuous mappings is strongly continuous.*

The following example shows that the result of Theorem 2.2 is not necessarily true for continuous mappings.

EXAMPLE 2.1. Let $X=\{a,b,c\}$ and let $\mathscr{T}$ be the indiscrete topology for X. Let $f: (X,\mathscr{T})\to(X,\mathscr{T})$ be the identity mapping. If $\mathscr{T}^*$ be the discrete topology for X and if $g: (X,\mathscr{T})\to(X,\mathscr{T}^*)$ be the identity mapping, then f is continuous but $g\circ f$ is not continuous.

THEOREM 2.3. *If $f: X\to Y$ is a weakly continuous mapping and $g: Y\to Z$ is strongly continuous, then $g\circ f: X\to Z$ is strongly continuous.*

PROOF. Let A be any subset of Z. Since g is strongly continuous, therefore

$g^{-1}(A)$ is an open as well as closed subset of Y. Since f is weakly continuous and $g^{-1}(A)$ is an open subset of Y, therefore $(f^{-1}(g^{-1}(A)))^{-}\subset f^{-1}(g^{-1}(A)^{-})=f^{-1}(g^{-1}(A))=(g\circ f)^{-1}(A)$. It follows that $(g\circ f)^{-1}(A)$ is a closed subset of X. Hence $g\circ f$ is strongly continuous.

COROLLARY 2.2. *If $f: X\to Y$ is continuous and $g: X\to Z$ is strongly continuous, then $g\circ f: X\to Z$ is strongly continuous.*

THEOREM 2.4. *Let $f: X\to \prod_{\alpha\in A} X_\alpha$ be a strongly continuous mapping. Let $f_\alpha: X\to X_\alpha$, for eash $\alpha\in A$ be defined as $f_\alpha(x)=x$ if $f(x)=(x_\alpha)$. Then the mapping f_α is strongly continuous for each $\alpha\in A$.*

PROOF. Let P_α denote the projection of $\prod X_\alpha$ onto X_α. Then, obviously, $f_\alpha=P_\alpha\circ f$ for each $\alpha\in A$. Since f is strongly continuous, therefore each f_α is strongly continuous in view of Theorem 2.2.

THEOREM 2.5. *Let $f_1: X_1\to Y_1$ and $f_2: X_2\to Y_2$ be strongly continuous mappings. Let $X=X_1\times X_2$ and $Y=Y_1\times Y_2$. Let $f: X\to Y$ be defined as $f(x_1, x_2)=(f_1(x_1), f_2(x_2))$. Then f is strongly continuous.*

PROOF. Let $y\in Y_1\times Y_2$. Then $y=(y_1, y_2)$ where $y_1\in Y_1$ and $y_2\in Y_2$. Then $f^{-1}(y)=f_1^{-1}(y_1)\times f_2^{-1}(y_2)$. Since $f_1: X_1\to Y_1$ is strongly continuous, therefore $f_1^{-1}(y_1)$ is an open subset of X_1. Similarly, $f_2^{-1}(y_2)$ is an open subset of X_2. It follows that $f^{-1}(y)$ is an open subset of X. Hence f is strongly continuous.

3. Connectedness and strong continuity

Levine [3] proved that if $f: X\to Y$ is strongly continuous and A is any non-empty connected subset of X, then $f(A)$ consists of a single point. He further proved that the converse of the above statement holds if X is locally connected. It follows immediately that every strongly continuous mapping $f: X\to Y$ is constant on every component of X. Cullen [2] proved that every strongly continuous mapping is constant on every quasi-component of X where by a quasi-component is meant a maximal quasi-connected set (A closed set A is said to be quasi-connected if for every clo-open set U such that $U\cap A\neq\phi$, we have $A\subset U$).

THEOREM 3.1. *A space X is connected if and only if every strongly continuous mapping on X is constant.*

PROOF. The 'only if' part is obvious in view of Theorem 2 of Levine [3]. To prove the 'if' part, let X be disconnected. Then there exists a non-empty proper subset A of X which is both open as well as closed. Let $Y=\{a,b\}$ where $a\neq b$ and let $\mathscr{T}$ be any topology for Y. Define a mapping $f: X\to Y$ such that $f(A)=\{a\}$ and $f(X-A)=\{b\}$. Obviously, f is a non-constant strongly continuous mapping on X. But this is a contradiction. Hence X must be connected.

THEOREM 3.2. *Let X be a locally connected space and let $\mathfrak{M}$ be the cardinality of the family $\mathscr{C}$ of all components of X. Then any space with cardinality $\leq\mathfrak{M}$ is the image of X under some strongly continuous mapping. But a space with cardinality $>\mathfrak{M}$ is not the image of X under any strongly continuous mapping.*

PROOF. Let Y be any space with cardinality $\mathfrak{N}\leq\mathfrak{M}$. Let $\mathscr{C}'$ be the subfamily of $\mathscr{C}$ of cardinality $\mathfrak{N}$. Then there is a one-to-one mapping g from $\mathscr{C}'$ to Y. Define a mapping $f: X\to Y$ such that $f(x)=g(D_x)$ when $x\in D_x\in\mathscr{C}'$ and $f(x)=g(D_0)$ when $x\in D\in\mathscr{C}$ but $D\in\mathscr{C}'$, D_0 being some fixed member of $\mathscr{C}'$. Since each $D\in\mathscr{C}$ is open as well as closed, therefore $f^{-1}(y)$ is open as well as closed for each $y\in Y$. Hence f is strongly continuous. Further if $f: X\to Y$ is a strongly continuous mapping of X onto Y then f can take at most $\mathfrak{M}$ different values where $\mathfrak{M}$ is the cardinality of $\mathscr{C}$. Therefore the cardinality of Y is $\leq\mathfrak{M}$.

DEFINITION 3.1. A space X is said to be *totally disconnected* if the singletons are the only connected subsets of X.

THEOREM 3.3. *A mapping $f: X\to Y$ from a locally connected space X to a totally disconnected space Y is strongly continuous if and only if f is a connected mapping.*

PROOF. Every strongly continuous mapping is obviously connected. Conversely, if f is connected, then it must be constant on every component of X, since in a totally disconnected space, singletons are the only connected sets. Since X is locally connected, therefore every component of X is open. If follows that $f^{-1}(y)$ is open for each $y\in Y$. Hence f is strongly continuous.

COROLLARY 3.1. *Every weakly continuous (and hence every continuous) mapping from a locally connected space to a totally disconnected space is strongly continuous.*

Obviously, if $f: (X,\mathscr{T})\to(Y,\mathscr{U})$ be a strongly continuous mapping then for any other topology $\mathscr{U}'$ for Y, the mapping $f: (X,\mathscr{T})\to(Y,\mathscr{U}')$ is also strongly continuous. It follows therefore that the knowledge of the domain of a strongly continuous mapping does not help in knowing the topology of its range. However, a study of all strongly continuous mappings on a space X may reveal interesting properties about the space X.

THEOREM 3.4. *Let $\mathscr{D}$ denote the decomposition of a space X into components. Let $\mathfrak{M}$=cardinality of $\mathscr{D}$. The space X is locally connected if and only if there exists a space Y of cardinality which is a strongly continuous image of X.*

PROOF. The 'only if' part can be proved as the first part of Theorem 3.2. To prove the 'if' part, suppose that the space Y described in the theorem exists and that $f:(X, \mathscr{T})\to(Y, \mathscr{U})$ is strongly continuous. Let D be any component of X. Then $f(D)=y_d$ where $y_d\in Y$. Since $f^{-1}(y_d)=f^{-1}(f(D))$ is closed as well as open, therefore either $f^{-1}(f(D))=D$ or there is another component D' of X such that $D'\subset f^{-1}(y_d)$. This completes the proof of the theorem.

THEOREM 3.5. *Let $f:A\to Y$ be a strongly continuous mapping. Then f can be extended strongly continuously to any locally connected space which contains A as a closed and open set.*

PROOF. Let X be any locally connected space containing A as a closed and open set. Define a mapping $g:X\to Y$ such that $g(x)=f(x)$ for all $x\in A$ and if $x\in X-A$ and K is the component of X containing x, then let $g(K)$=some fixed point of Y (A being both open and closed, $K\cap A=\phi$). Since g is a mapping from a locally connected space such that the image of every non-empty connected set is a single point, therefore g is strongly continuous.

DEFINITION 3.2. A space X is said to be *semi-locally connected* if for every point $x\in X$ and every neighbourhood U of x, there exists a neighbourhood V of x such that $x\in V\subset U$ and $X-V$ consists of a finite number of component.

THEOREM 3.6. *Let X be a semi-locally connected space and let $f:X\to Y$ be a strongly continuous mapping from X onto Y. Then Y must be a finite space.*

PROOF. Let U be any open subset of X such that $X-U$ consists of a finite number of components. In view of Theorem 2 of Levine [3], $f(X-U)$ consists of a finite number of points only. Let $f(X-U)=F$. Then $f^{-1}(Y-F)\subset U$ and $f^{-1}(Y-F)$ is closed. It follows that $f^{-1}(F)$ is open. Therefore, we can find an open subset G of $f^{-1}(F)$ such that $X-G$ has a finite number of components. Therefore $f(X-G)$ has a finite number of points. Now, $X-f^{-1}(F)\subset X-G$ and $f(X-G)$ is finite. It follows that $Y-F$ is a finite set. Hence $F\cup(Y-F)$ is a finite set.

DEFINITION 3.3. A space X is said to be *weakly locally connected* if every point of X has a connected neighbourhood.

THEOREM 3.7. *Let $f: X \to Y$ be a mapping from a weakly locally connected space X to a space Y such that the image of every non-empty connected subset of X is a single point. Then f is strongly continuous.*

PROOF. Let A be any subset of X. We shall show that $f(A') \subset f(A)$ where A' denotes the derived set of A. Let $x \in A'$. Since X is weakly locally connected, therefore there exists a connected neighbourhood N of x. Since $x \in A'$, therefore $N \cap A \neq \phi$. Obviously $f(x) \in f(N)$. Since $f(A \cap N) \subset f(N)$ and N is connected, therefore $f(N)$ is a single point. It follows that $f(A \cap N) = f(x)$. Since $f(A \cap N) \subset f(A)$, therefore $f(x) \in f(A)$. Thus $f(A') \subset f(A)$ and hence f is strongly continuous.

COROLLARY 3.2. [Levine, 3]. *If $f: X \to Y$ is a mapping from a locally connected space X to a space Y such that the image of every non-empty connected subset of X is a single point, then f is strongly continuous.*

With the help of Theorem 3.7, the following stronger form of Theorem 3.3 can be proved.

THEOREM 3.8. *A mapping $f: X \to Y$ from a weakly locally connected space X to a totally disconnected space Y is strongly continuous if and only if f is a connected mapping.*

4. Compact mappings and strong continuity

DEFINITION 4.1. A mapping $f: X \to Y$ is said to be a *compact mapping* if the inverse image of every compact subset of Y is a compact subset of X.

THEOREM 4.1. *Every strongly continuous mapping from a compact space is compact.*

PROOF. Let $f: X \to Y$ be a strongly continuous mapping where X is a compact space. Let A be any compact subset of Y. Since f is strongly continuous, therefore $f^{-1}(A)$ is a closed subset of X. Since X is compact, therefore $f^{-1}(A)$ is compact. Hence f is a compact mapping.

DEFINITION 4.2. [Levine, 5]. A space X is said to be a *C-C space* if a set is compact if and only if it is closed.

THEOREM 4.2. *If $f: X \to Y$ is a mapping from a C-C space X to a hereditarily compact space Y (that is, every subspace of Y is compact) then f is strongly continuous if and only if it is compact.*

PROOF. If f is strongly continuous and A is any compact subset of Y, then $f^{-1}(A)$ is a closed subset of X. Since X is a C-C space, therefore $f^{-1}(A)$ is compact and hence f is a compact mapping. Conversely, if f is compact and if B be any subset of Y, then Y being hereditarily compact, B is compact. Therefore $f^{-1}(B)$ is a compact subset of X. Since X is a C-C space, therefore $f^{-1}(B)$ is closed. Hence f is strongly continuous.

THEOREM 4.3. *If $f: X \to Y$ is a strongly continuous mapping, then the image of every compact subset of X is a finite set.*

PROOF. Let A be a compact subset of X. Since f is strongly continuous, therefore $f^{-1}(y)$ is open for each $y \in Y$. Thus the family $\{f^{-1}(y) : y \in f(A)\}$ is an open covering of the compact set A. Therefore there exist finitely many points $y_1, \cdots, y_n$ in $f(A)$ such that $A \subset \cup \{f^{-1}(y_i) : i=1, \cdots, n\}$. If $f(A)$ is infinite, then there exists a point $z \in f(A)$ such that $z \neq y_i$ for any $i=1, \cdots, n$. This means that there exists $x \in A : f(x)=z$. It follows that $x \notin \cup \{f^{-1}(y_i) : i=1, \cdots, n\}$. But this is a contradiction. Hence $f(A)$ must be finite.

5. Some preservation results

DEFINITION 5.1. A space X is said to be *almost compact* if every open covering of X has a finite subfamily whose closures cover X.

THEOREM 5.1. *Every strongly continuous image of an almost compact space is compact.*

PROOF. Let $f: X \to Y$ be a strongly continuous mapping of an almost compact space X onto a space Y. If $\{U_\alpha : \alpha \in \Lambda\}$ be any open covering of Y, then $\{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a covering of X by clo-open sets. Since X is almost compact, therefore there exists a finite subfamily $\{f^{-1}(U_{\alpha_i}) : i=1, \cdots, n\}$ of $\{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ which covers X. It follows that $\{U_{\alpha_i} : i=1, \cdots, n\}$ is a finite subcovering of $\{U_\alpha : \alpha \in \Lambda\}$ and hence Y is compact.

COROLLARY 5.1. *Every strongly continuous image of a nearly compact space, that is, every covering by regularly open sets has a finite subcovering, is compact.*

DEFINITION 5.2. [Singal and Arya, 8]. A space X is said to be *nearly paracompact* if for every open covering $\mathscr{U}$ of X, there exists a locally finite family $\mathscr{V}$ of open subsets of X such that each member of $\mathscr{V}$ is contained in some member of $\mathscr{U}$ and the family $\{V^0 : V\in\mathscr{V}\}$ covers X.

DEFINITION 5.3. [Arhangel'skii, 1]. A mapping f of X onto Y is said to be *almost open* if for each $y\in Y$, there exists an $x\in f^{-1}(y)$ such that there is an open base $\mathscr{G}$ at x such that $f(G)$ is open for each $G\in\mathscr{G}$.

THEOREM 5.2. *If f is a closed, strongly continuous, almost open mapping of X onto Y such that $f^{-1}(y)$ is compact for each $y\in Y$, then Y is paracompact if X is nearly paracompact.*

PROOF. Let $\{U_\alpha : \alpha\in\Lambda\}$ be any open covering of Y. Then $\{f^{-1}(U_\alpha) : \alpha\in\Lambda\}$ is an open covering of X. Since X is nearly paracompact, therefore there exists a locally finite family $\{V_\beta : \beta\in\Lambda\}$ of open subsets of X such that each V_β is contained in some $f^{-1}(U_\alpha)$ and the family $\{\overline{V}_\beta^0 : \beta\in\Lambda\}$ is a covering of X. We shall prove that the family $\{[f(V_\beta)]^0 : B\in\Lambda\}$ is a locally finite open refinement of $\{U_\alpha : \alpha\in\Lambda\}$. Let $y\in Y$. Then, for each $x\in f^{-1}(y)$, there exists an open set M_x such that $x\in M_x$ and M_x intersects atmost finitely many members of $\{V_\beta : \beta\in\Lambda\}$. Then since $\{M_x : x\in f^{-1}(y)\}$ is an open covering of $f^{-1}(y)$ and $f^{-1}(y)$ is compact therefore there exist finitely many sets M_{x_i}, $i=1, \cdots, n$ such that $f^{-1}(y)\subset \bigcup_{i=1}^{n} M_{x_i}$. Since each M_{x_i} intersects finitely many sets V_β therefore the set $M=\bigcup_{i=1}^{n} M_{x_i}$ intersects finitely many V_β's. It follows that the set $Y-f(X-M)$ is an open set containing y which intersects atmost finitely many sets in $\{f(V_\beta) : \beta\in\Lambda\}$. Therefore, the family $\{f(V_\beta) : \beta\in\Lambda\}$ and hence also the family $\{[f(V_\beta)]^0 : \beta\in\Lambda\}$ is locally finite. Since $\{\overline{V}_\beta^0 : \beta\in\Lambda\}$ is an open covering of X and f is almost open, therefore $\{[f(\overline{V}_\beta^0)]^0 : \beta\in\Lambda\}$ is a covering of Y [1]. But $[f(\overline{V}_\beta^0)]^0\subset[f(\overline{V}_\beta)]^0 \subset[f(V_\beta)]^0$. Therefore $\{[f(V_\beta)]^0 : \beta\in\Lambda\}$ is also a covering of Y. Obviously, each $[f(V_\beta)]^0$ is open and is contained in some U_α. Hence $\{[f(V_\beta)]^0 : \beta\in\Lambda\}$ is a locally finite open refinement of $\{U_\alpha : \alpha\in\Lambda\}$ and so Y is paracompact.

DEFINITION 5.9. [Singal and Arya, 7]. A space X is said to be *almost paracompact* if for every open covering $\mathscr{U}$ of X, there exists a locally finite family $\mathscr{V}$ of open subsets of X such that each member of $\mathscr{V}$ is contained in some member of $\mathscr{U}$ and the family $\{V : \overline{V}\in\mathscr{V}\}$ is a covering of X.

THEOREM 5.3. *If f is a strongly continuous, open mapping of a space X onto a space Y such that $f^{-1}(y)$ is compact for each $y \in Y$, then Y is pointwise paracompact if X is almost paracompact.*

PROOF. Let $\{U_\alpha : \alpha \in \Lambda\}$ be any open covering Y. Then $\{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is an open covering of X. Since X is almost paracompact, there exists a locally finite family $\{V_\beta : \beta \in \Lambda\}$ of open subsets of X such that the family $\{\bar{V}_\beta : \beta \in \Lambda\}$ covers X. Since $\{V_\beta : \beta \in \Lambda\}$ is locally finite and $f^{-1}(y)$ is compact for each $y \in Y$, therefore it is easy to verify that $\{f(V_\beta) : \beta \in \Lambda\}$ is point-finite. Since f is open, therefore each $f(V_\beta)$ is open. Also, f is strongly continuous and therefore $\overline{f(V_\beta)} = f(\bar{V}_\beta)$. It follows that $\{f(V_\beta) : \beta \in \Lambda\}$ covers Y. Thus $\{f(\bar{V}_\beta) : \beta \in \Lambda\}$ is a point-finite open refinement of $\{U_\alpha : \alpha \in \Lambda\}$ and hence Y is pointwise paracompact.

6. Completely continuous mappings

DEFINITION 6.1. A mapping $f : X \to Y$ is said to be *completely continuous* if the inverse image of every open subset of Y is a regularly open subset of X.

Obviously, every strongly continuous mapping is completely continuous and every completely continuous mapping is continuous. The converse implications do not hold as is shown by the following example.

EXAMPLE 6.1. Let $X = \{a, b, c, d\}$ and let $\mathscr{T} = \{X, \phi, \{a, b, c\}, \{c\}, \{a, b\}\}$. Let $Y = \{p, q, r\}$ and let $\mathscr{U} = \{X, \phi, \{p\}, \{q\}, \{p, q\}\}$ If $f : (X, \mathscr{T}) \to (Y, \mathscr{U})$ be the mapping defined by $f(a) = p$, $f(b) = p$, $f(c) = f(d) = r$, then f is a completely continuous mapping which is not strongly continuous.

EXAMPLE 6.2. Let $X = \{a, b, c, d\}$ and let$= \{X, \phi, \{a, b\}\}$ Let $Y = \{p, q\}$ and let $\mathscr{U} = \{X, \phi, \{p\}\}$. If $f : (X, \mathscr{T}) \to (Y, \mathscr{U})$ be the mapping defined by $f(a) = p = f(b)$ and $f(c) = f(d) = q$, then f is a continuous mapping which is not completely continuous.

The restriction of a completely continuous mapping may fail to be completely continuous as is shown by the following example.

EXAMPLE 6.3. Let $X = \{a, b, c, d\}$ and let $\mathscr{T} = \{X, \phi, \{a, b\}, \{c\}, \{a, b, c\}\}$. Let $Y = \{x, y, z\}$ and let $\mathscr{U} = \{Y, \phi, \{x, y\}\}$. If $f : (X, \mathscr{T}) \to (Y, \mathscr{U})$ be the mapping defined by $f(a) = x$, $f(b) = y$, $f(c) = f(d) = z$, then f is completely continuous. However, the restriction of f to the set $\{a, d\}$ is not completely continuous.

THEOREM 6.1. *If $f : X \to Y$ is completely continuous and $g : Y \to Z$ is continuous,* *then $g\circ f: X \to Z$ is completely continuous.*

PROOF. Let U be any open subset of Z. Since g is continuous, therefore $g^{-1}(U)$ is an open subset of Y. Since f is completely continuous, therefore $f^{-1}(g^{-1}(U))$ is a regularly open subset of X. That is, $(g\circ f)^{-1}(U)$ is a regularly open subset of X. Hence $g\circ f$ is completely continuous.

COROLLARY 6.1. *The composite of two completely continuous mappings is completely continuous.*

DEFINITION 6.2. [Singal and Singal, 10] A mapping $f: X \to Y$ is said to be *almost open* if the image of every regularly open set is open.

THEOREM 6.2. *If $f: X \xrightarrow{onto} Y$ is almost open and completely continuous and $g: Y \to Z$ is a mapping such that $g\circ f$ is completely continuous, then g is continuous.*

PROOF. Let G be any open subset of Z. Since $g\circ f$ is completely continuous, therefore $(g\circ f)^{-1}(G)$ is a regularly open subset of X. Since f is almost open, therefore $f((g\circ f)^{-1}(G))$ is an open subset of Y, that is, $f(f^{-1}(g^{-1}(G)) = g^{-1}(G)$ is an open subset of Y. Hence, g is continuous.

THEOREM 6.3. *Let $f: X \to \prod_{\alpha\in\Lambda} X_\alpha$ be completely continuous. For each $\alpha\in\Lambda$, define $f_\alpha: X \to X_\alpha$ by setting $f_\alpha(x) = x_\alpha$ where $f(x) = (x_\alpha)$. Then each f_α is completely continuous.*

PROOF. Let p_α denote the projection of X onto X_α. Then for each $\alpha\in\Lambda$, $f_\alpha = p_\alpha \circ f$. Since f is completely continuous and p_α is continuous, therefore f is completely continuous in view of theorem 6.1.

THEOREM 6.4. *Every completely continuous image of a nearly compact space is compact.*

PROOF. Let $f: X \to Y$ be a completely continuous mapping of a nearly compact space X onto a space Y. Let $\{U_\alpha : \alpha\in\Lambda\}$ be any open covering of Y. Thus, $\{f^{-1}(U_\alpha) : \alpha\in\Lambda\}$ is a regular open covering of X. There exists therefore a finite subcover $\{f^{-1}(U_{\alpha_i}) : i=1, 2, \cdots, n\}$ of $\{f^{-1}(U_\alpha) : \alpha\in\Lambda\}$. It follows that $\{U_{\alpha_i} : i=1, 2, \cdots, n\}$ is a finite subcover of $\{U_\alpha : \alpha\in\Lambda\}$.

THEOREM 6.5. *Let f be a closed, completely continuous, almost-open mapping of a space X onto a space Y such that $f^{-1}(y)$ is compact for each $y\in Y$. Then if X is nearly paracompact, then Y is paracompact.*

PROOF. Let $\{U_\alpha : \alpha \in \Lambda\}$ be any open covering of Y. Then $\{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a regularly open covering of X. Since X is nearly paracompact, therefore there exists a locally finite regular open refinement $\{V_\beta : \beta \in \Lambda\}$ of $\{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$. Consider the family $\{f(V_\beta) : \beta \in \Lambda\}$. Since f is almost open, therefore each $f(V_\beta)$ is open. Also, since f is closed such that $f^{-1}(y)$ is compact for each $y \in Y$, therefore $\{f(V_\beta) : \beta \in \Lambda\}$ is locally finite. Thus $\{f(V_\beta) : \beta \in \Lambda\}$ is a locally finite open refinement of $\{U_\alpha : \alpha \in \Lambda\}$ and hence Y is paracompact.

THEOREM 6.6. *Let f be a completely continuous, open mapping of a nearly paracompact space X onto a space Y such that $f^{-1}(y)$ is compact for each $y \in Y$. Then Y is pointwise paracompact.*

PROOF. Let $\{U_\alpha : \alpha \in \Lambda\}$ be any open covering of Y. Then $\{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a regular open covering of X. Since X is nearly paracompact, therefore there exists a locally finite open refinement $\{V_\beta : \beta \in \Lambda\}$ of $\{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$. It is easy to verify now that $\{f(V_\beta) : \beta \in \Lambda\}$ is a point-finite open refinement of $\{U_\alpha : \alpha \in \Lambda\}$ Hence Y is pointwise paracompact.

DEFINITION 6.3. [Singal and Singal, 11]. A space X is said to be *mildly normal* if every pair of disjoint regularly closed subsets of X can be strongly separated.

THEOREM 6.7. *If $f : X \to Y$ be a completely continuous closed mapping of a mildly normal space X onto a space Y, then Y is normal.*

PROOF. Let A and B be any two disjoint closed subsets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint regularly closed subsets of X. Since X is mildly normal, therefore there exist disjoint open sets O_A and O_B such that $f^{-1}(A) \subset O_A$ and $f^{-1}(B) \subset O_B$. Let $G_A = \{y : f^{-1}(y) \subset O_A\}$ and $G_B = \{y : f^{-1}(y) \subset O_B\}$. Then $G_A = Y - f(X - O_A)$ and $G_B = Y - f(X - O_B)$. It follows that G_A and G_B are disjoint open sets containing A and B respectively. Hence Y is normal.

DEFINITION 6.4. [Singal and Arya, 9]. A space X is said to be *almost-regular* if for every regularly closed set F and a point $x \notin F$, there exist open set U and V such that $x \in U$, $F \subset V$ and $U \cap V = \phi$.

THEOREM 6.8. *If f is a completely continuous, closed mapping of an almost regular space X onto a space Y such that $f^{-1}(y)$ is compact for each $y \in Y$, then Y is regular.*

PROOF. Let F be a closed subset of Y and let $y \notin F$. Then $f^{-1}(F)$ is a regularly closed subset of X such that $f^{-1}(y) \cap f^{-1}(F) = \phi$. For each $x \in f^{-1}(F)$, since $f^{-1}(F)$ is a regularly closed set not containing x, therefore there exist disjoint open sets U_x and V_x such that $x \in U_x$ and $f^{-1}(F) \subset V_x$. Then $\{U_x : x \in f^{-1}(y)\}$ is an open covering of $f^{-1}(y)$. Since $f^{-1}(y)$ is compact, therefore there exists a finite subcover $\{U_{x_i} : i=1, 2, \cdots, n\}$ of $\{U_x : x \in f^{-1}(y)\}$. If $U = \bigcup_{i=1}^{n} U_{x_i}$ and $V = \bigcup_{i=1}^{n} V_{x_i}$, then U and V are disjoint open sets containing $f^{-1}(y)$ and $f^{-1}(F)$ respectively. If $M = Y - f(X-U)$ and $N = Y - f(X-V)$, then M and N are disjoint open sets such that $y \in M$ and $F \subset N$. Hence Y is regular.

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