Kyungpook Math. J.
Volume 14, Number 1
June, 1974

A NOTE ON BLOCK CIRCULANT MATRICES

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The purpose of this note is to present a simple proof for the main theorem in [3]. Our method is similar to the one used in [1].

Let $C=(c_{ij})$ be a $n\times n$ circulant with c_{ij} belonging to the complex number field K, and $i,j=0, 1, \cdots, n-1$. Let $f(x)=\sum_{j=0}^{n-1} c_{0j}x^j$. It is well known that the eigenvalues of C are $\mu_k=f(\omega^k)$, $k=0, 1, \cdots, n-1$, and the eigenvector corresponding to each μ_k is a column vector $\{\omega^0, \omega^k, \omega^{2k}, \cdots, \omega^{(n-1)k}\}$ for $k=0, 1, \cdots, n-1$ where $\omega=\exp\{2\pi i/n\}$ (in fact, ω can be any primitive n-th root of unity). Let $P=(p_{ij})=(\omega^{ij})$, $i, j=0, 1, \cdots, n-1$, $\Omega=(1/\sqrt{n})P$. Then P is a Vandermonde matrix and Ω is a unitary matrix.

Also,

$$\Omega^{-1}C\Omega=\bar{\Omega}'C\Omega=\text{diag}\ \{\mu_0, \mu_1, \cdots, \mu_{n-1}\}$$

where diag $\{\mu_0, \mu_1, \cdots, \mu_{n-1}\}$ denotes the diagonal matrix with $\mu_0, \mu_1, \cdots, \mu_{n-1}$ on the diagonal, and Ω' and $\bar{\Omega}$ denote the transpose and complex conjugate of Ω respectively.

The main theorem (Theorem 5) in [3] states as follows: *Let*

$$A=\begin{bmatrix} A_0 & A_1 & \cdots & A_{m-1} \\ A_{m-1} & A_0 & \cdots & A_{m-2} \\ \vdots & \vdots & & \vdots \\ A_1 & A_2 & \cdots & A_0 \end{bmatrix}$$

be a $m\times m$ block circulant with each A_i being a $n\times n$ circulant matrix. Let $P=(p_{ij})=(\omega^{ij})$ be the $n\times n$ matrix as before. Let $r_0, r_1, \cdots, r_{m-1}$ be the m-th roots of unity. If Q is given by the following matrix:

*This work was done while the author was at Carnegie-Mellon University (under a faculty exchange program between Carnegie-Mellon University and the University of Pittsburgh). The author wishes to thank Professor R. J. Duffin for helpful discussions.

$$Q=\begin{bmatrix} p & p & \cdots & p \\ r_0 p & r_1 p & \cdots & r_{m-1}p \\ \vdots & \vdots & \cdots & \vdots \\ r_0^{m-1}p & r_1^{m-1}p & \cdots & r_{m-1}^{m-1}p \end{bmatrix}$$

*We have $Q^{-1}AQ=D$ with D being a matrix of diagonal blocks D_0, D_1, $\cdots$, D_{m-1} where each D_i is diagonal. The diagonal elements are given by the eigenvalues of the matrix $\sum_{k=0}^{m-1} A_k r_i^k$. Moreover, given any $nm\times nm$ diagonal block matrix $D=$*diag *$\{D_0, D_1, \cdots, D_{m-1}\}$ where each D_i is a $n\times n$ diagonal matrix, $A=QDQ^{-1}$ is a block circulant with each block being a circulant matrix.*

(We note that r_0, r_1, $\cdots$, r_{m-1} are m-th roots of unity implying $r_j=r^j$, $j=0, 1, \cdots, m-1$ where $r=\exp\{2\pi i/m\}$).

In [3], a proof for three block case ($m=3$) is given, and it states that the proof for the general case is omitted since it is just an extension of the three block case. Here we present a simple proof for the general case by using elementary properties of Kronecker product of matrices.

The proof goes as follows: The matrix A is equal to

$$I\otimes A_0+T\otimes A_1+T^2\otimes A_2+\cdots+T^{m-1}\otimes A_{m-1}$$

where T is the $m\times m$ permutation matrix corresponding to the permutation

$$\begin{pmatrix} 0 & 1 & \cdots & i & \cdots & m-1 \\ 1 & 2 & \cdots & i+1 & \cdots & 0 \end{pmatrix}$$

$I=T^m$ is the identity matrix and $\otimes$ denotes the Kronecker product. Clearly, each T^k, $k=0, 1, \cdots, m-1$, is a $m\times m$ circulant.

Let $R=(s_{ij})=(r^{ij})$ be a $m\times m$ matrix with $r=\exp\{2\pi i/m\}$ and $\Gamma=(1/\sqrt{m})R$. Then, again, R is a $m\times m$ Vandermonde matrix and Γ is a unitary matrix. By using elementary properties of Kronecker product of matrices (e.g., see pp. 68–70 in [2]), we have

$$\begin{aligned} &(\Gamma\otimes\Omega)^{-1}A(\Gamma\otimes\Omega) \\ &=(\Gamma^{-1}\otimes\Omega^{-1})(I\otimes A_0+T\otimes A_1+\cdots+T^{m-1}\otimes A_{m-1})(\Gamma\otimes\Omega) \\ &=(I\otimes\Omega^{-1}A_0\Omega)+(\Gamma^{-1}T\Gamma\otimes\Omega^{-1}A_1\Omega)+\cdots+(\Gamma^{-1}T^{m-1}\Gamma\otimes\Omega^{-1}A_{m-1}\Omega) \qquad (1) \end{aligned}$$

Since each A_k is a $n\times n$ circulant, $\Omega^{-1}A_k\Omega$ is a diagonal matrix, denoted by

E_k, with eigenvalues of A_k on the diagonal for $k=0, 1, \cdots, n-1$. Since each T^j is a $m\times m$ permutation matrix, $\Gamma^{-1}T^j\Gamma=\text{diag}\ \{r^0, r^j, r^{2j}, \cdots, r^{(m-1)j}\}$ for $j=0, 1, \cdots, m-1$. Thus, (1) is equal to, i.e.,

$$\begin{aligned}&(\Gamma\otimes\Omega)^{-1}A(\Gamma\otimes\Omega)\\&=I\otimes E_0+\text{diag}\{r^0, r^1, \cdots, r^{m-1}\}\otimes E_1+\cdots+\text{diag}\{r^0, r^{m-1}, \cdots, r^{(m-1)(m-1)}\}\otimes E_{m-1}\\&=\text{diag}\{\sum_{k=0}^{m-1} E_k, \sum_{k=0}^{m-1} r^k E_k, \sum_{k=0}^{m-1} r^{2k}E_k, \cdots, \sum_{k=0}^{m-1} r^{(m-1)k}E_k\}.\end{aligned}$$

This means that the i-th diagonal element is a diagonal matrix denoted by D_i, and the diagonal elements of D_i are given by the eigenvalues of the matrix $\sum_{k=0}^{m-1} A_k r^{ik}$.

Now we show that $A=(\Gamma\otimes\Omega)D(\Gamma\otimes\Omega)^{-1}$ is a block circulant with each block being a circulant matrix and $D=\text{diag}\{D_0, D_1, \cdots, D_{m-1}\}$ where each D_i is a $n\times n$ diagonal matrix. We need the following:

LEMMA. *If* $D_p=\text{diag}\ \{d^p_{00}, d^p_{11}, \cdots, d^p_{(n-1)(n-1)}\}$ *then* $F_p=\Omega D_p\Omega^{-1}$ *is a circulant for* $p=0, 1, \cdots, m-1$.

PROOF. $(\Omega D_p\ \Omega^{-1})_{ij}=\frac{1}{n}\sum_{k=0}^{n-1}\omega^{ik}d^p_{kk}\bar{\omega}^{jk}$, and

$$(\Omega D_p\Omega^{-1})_{(i+1)(j+1)}=\frac{1}{n}\sum_{k=0}^{n-1}\omega^{(i+1)k}d^p_{kk}\bar{\omega}^{(j+1)k}$$

since $\omega\bar{\omega}=1$, $(\Omega D_p\Omega^{-1})_{ij}=(\Omega D_p\Omega^{-1})_{(i+1)(j+1)}$ for $i, j=0, 1, \cdots, n-1$ (the subscripts are taken modulo n).

Hence each $F_p=\Omega D_p\Omega^{-1}$ is a circulant for $p=0, 1, \cdots, m-1$.

Now, by using our Lemma,

$$\begin{aligned}A&=(\Gamma\otimes\Omega)D(\Gamma\otimes\Omega)^{-1}\\&=(\Gamma\otimes\Omega)\ (\text{diag}\ \{D_0, D_1, \cdots, D_{m-1}\})\ (\Gamma^{-1}\otimes\Omega^{-1})\\&=(\Gamma\otimes\Omega)\ [\text{diag}\{1, 0, \cdots, 0\}\otimes D_0+\text{diag}\ \{0, 1, 0, \cdots, 0\}\otimes D_1+\cdots\\&\qquad+\text{diag}\{0, 0, \cdots, 0, 1\}\otimes D_{m-1}](\Gamma^{-1}\otimes\Omega^{-1})\\&=\Gamma(\text{diag}\{1, 0, \cdots, 0\})\Gamma^{-1}\otimes F_0+\Gamma\ (\text{diag}\ \{0, 1, 0, \cdots, 0\})\Gamma^{-1}\otimes F_1+\cdots\\&\qquad+\Gamma(\text{diag}\{0, \cdots, 0, 1\})\Gamma^{-1}\otimes F_{m-1}.\end{aligned}\tag{2}$$

Hence, by using (2), the ij entry in the matrix A is

$$\frac{1}{m}(r^{i0}{}_1\bar{r}^{j0}F_0+r^{i1}{}_1\bar{r}^{j1}F_1+\cdots+r^{i(m-1)}{}_1\bar{r}^{j(m-1)}F_{m-1}),$$

and the $(i+1)$ $(j+1)$ entry in the matrix A is

$$\frac{1}{m}(r^{(i+1)0}{}_1\bar{r}^{(j+1)0}F_0+r^{(i+1)1}{}_1\bar{r}^{(j+1)1}F_1+\cdots+r^{(i+1)(m-1)}{}_1\bar{r}^{(j+1)(m-1)}F_{m-1})$$

Since $r\bar{r}=1$, the ij entry in A is equal to the $(i+1)$ $(j+1)$ entry in A for $i,j=0, 1, \cdots, m-1$ (all the ij entries are taken modulo m). Hence, A is a block circulant.

The matrix Q in the theorem is our $R\otimes P$.

Then $Q^{-1}=\frac{1}{mn}(R^{-1}\otimes P^{-1})$, and

$$\begin{aligned} Q^{-1}AQ&=\frac{1}{mn}(R^{-1}\otimes P^{-1})A(R\otimes P) \\ &=\left(\frac{1}{\sqrt{mn}}(R^{-1}\otimes P^{-1})\right)A\left(\frac{1}{\sqrt{mn}}(R\otimes P)\right) \\ &=(\Gamma\otimes\Omega)^{-1}A(\Gamma\otimes\Omega). \end{aligned}$$

REMARKS. It is indicated on page 19 in [3] that R and P are Vandermonde matrices and the determinant of a Vandermonde matrix is well known. However, Q is not a Vandermonde matrix. But the determinant of Q, $\det Q$, is equal to $\det(R\otimes P)=(\det R)^n(\det P)^m$ (see p. 70 in [2]). G. Trapp pointed out to me that, using the same method, one can deal with a block circulant each whose entry is again a block circulant, $\cdots$, etc.

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