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A NOTE ON A RIEMANNIAN SPACE WITH SASAKI-KILLING STRUCTURE

By Seiichi Yamaguchi and Fumizô Matsumoto

§ 0. Introduction.

T. Kashiwada [2] has dealt with Sasakian 3-structure introduced by Y. Y. Kuo [3] and obtained the following:

THEOREM A. *A Riemannian space with Sasakian 3-structure is an Einstein space.*

On the other hand, recently, I. Satô [4] has introduced the notion of a Riemannian space with Sasaki-Killing structure or special Sasaki-Killing structure and proved the following:

THEOREM B. *A Riemannian space with special Sasaki-Killing strucutre is an Einstein space.*

The purpose of this paper is to prove the following:

THEOREM. *A compact Riemannian space with Sasaki-Killing structure is an Einstein space. Then the Sasaki-Killing structure reduces to special.*

§ 1. Preliminaries.

Let M be an n-dimensional Riemannian space with metric tensor $g_{ij}(i, j, \cdots\cdots =1, 2, \cdots, n)$ and local coordinate systems $\{x^i\}$.

First, we recall the definitions of a special Killing p-form and a conformal Killing p-form. A p-form u with coefficients $u_{i_1\cdots i_p}$ is called a Killing p-form if it satisfies

$$\nabla_{i_0} u_{i_1\cdots i_p} + \nabla_{i_1} u_{i_0 i_2\cdots i_p} = 0,$$

where ∇ denotes the operator of covariant derivative with respect to the Riemannian connection. If a Killing p-form u satisfies

$$(1.1) \qquad \nabla_a \nabla_b u_{i_1\cdots i_p} + \beta\Big(g_{ab} u_{i_1\cdots i_p} + \sum_{\alpha=1}^{p} (-1)^\alpha g_{a i_\alpha} u_{b i_1\cdots \hat{i}_\alpha \cdots i_p}\Big) = 0,$$

where β is a nonzero constant and $\hat{i}_\alpha$ means that i_α is ommited, then it is called a special Killing p-form with constant β.

Moreover, we call a p-form w with coefficients $w_{i_1\cdots i_p}$ a conformal Killing

p-form, if there exists a $(p-1)$-form θ such that

(1.2) $$\nabla_{i_0} w_{i_1\cdots i_p} + \nabla_{i_1} w_{i_0\cdots i_p} = 2g_{i_0 i_1}\theta_{i_2\cdots i_p} - \sum_{\alpha=2}^{p}(-1)^\alpha (g_{i_0 i_\alpha}\theta_{i_1\cdots \hat{i}_\alpha \cdots i_p} + g_{i_1 i_\alpha}\theta_{i_0 i_2\cdots \hat{i}_\alpha\cdots i_p}).$$

Then we get

(1.3) $$\nabla^r w_{r i_2\cdots i_p} = (n-p+1)\theta_{i_2\cdots i_p}.$$

The form θ is called the associated form of w.

By virtue of (1.2), we obtain for a conformal Killing p-form w

(1.4) $$(dw)_{i_0 i_1\cdots i_p} = (p+1)(\nabla_{i_0} w_{i_1\cdots i_p} + \sum_{\alpha=1}^{p}(-1)^\alpha g_{i_0 i_\alpha}\theta_{i_1\cdots \hat{i}_\alpha\cdots i_p})$$

where d denotes the exterior differential operator.

The following theorem is known [1]:

THEOREM C. *In a compact orientable Riemannian space M, the following integral formula is valid for any p-form u*

(1.5) $$\int_M \{u^{i_1\cdots i_p}(\nabla^r\nabla_r u_{i_1\cdots i_p} + R_{i_1}{}^r u_{r i_2\cdots i_p} - \sum_{\alpha=2}^{p} R^a{}_{i_1 i_\alpha}{}^b u_{a i_2\cdots \overset{\alpha}{\hat{b}}\cdots i_p} + (n-p-1)\nabla_{i_1}\theta_{i_2\cdots i_p} + \sum_{\alpha=2}^{p}(-1)^\alpha \nabla_{i_\alpha}\theta_{i_1\cdots\hat{i}_\alpha\cdots i_p}) + (1/2)A_{i_0\cdots i_p}A^{i_0\cdots i_p}\}\,dV = 0,$$

where dV means the volume element of M and $A_{i_0\cdots i_p}$ and $\theta_{i_2\cdots i_p}$ are given by

(1.6) $$A_{i_0\cdots i_p} = \nabla_{i_0} u_{i_1\cdots i_p} + \nabla_{i_1} u_{i_0 i_2\cdots i_p} - 2g_{i_0 i_1}\theta_{i_2\cdots i_p} + \sum_{\alpha=2}^{p}(-1)^\alpha (g_{i_0 i_\alpha}\theta_{i_1\cdots\hat{i}_\alpha\cdots i_p} + g_{i_1 i_\alpha}\theta_{i_0 i_2\cdots\hat{i}_\alpha\cdots i_p}),$$

(1.7) $$(n-p+1)\theta_{i_2\cdots i_p} = \nabla^r u_{r i_2\cdots i_p}.$$

If the tensor field $A_{i_0\cdots i_p}$ vanishes identically, then u is a conformal Killing p-form.

§ 2. A Riemannian space with Sasaki-Killing structure.

Now, let us recall the definition and equations of a Riemannian space with Sasaki-Killing structure.

If a Riemannian space M admits a Sasakian structure $(\varphi_i^j, \xi^j, \eta_i, g_{ij})$ and an another almost contact metric structure $(\phi_i^j, \xi^j, \eta_i, g_{ij})$ having the following properties:

(2.1) the 2-form $\phi\left(=\frac{1}{2}\phi_{ij}dx^i\wedge dx^j\right)$ is a Killing form,

(2.2) $\varphi_i^{\ j}$ and $\phi_i^{\ j}$ satisfy $\varphi_r^{\ i}\phi_j^{\ r}+\phi_r^{\ i}\varphi_j^{\ r}=0$,

then M is said to have a Sasaki-Killing structure and a space with such a structure is called an SK-space.

Now, we define a tensor field $\theta_i^{\ j}=\varphi_r^{\ j}\phi_i^{\ r}$, then $(\theta_i^{\ j}, \xi^j, \eta_i, g_{ij})$ is also an almost contact metric structure.

In an SK-space, if a Killing 2-form ϕ is special with constant $\beta(\neq 0)$, then such an SK-space is called a special SK-space and β is known to be 1.

In an SK-space, the following equations are known [4]:

(2.3) $\nabla_j\nabla_i\phi_{lh}=(1/2)(R_{sjil}\phi_h^{\ s}+R_{sjlh}\phi_i^{\ s}+R_{sjhi}\phi_l^{\ s})$

(2.4) $\nabla^r\nabla_r\phi_{ji}=R_{jr}\phi_i^{\ r}+\phi_{ji}$,

(2.5) $(1/2)\phi^{rs}R_{rsth}\phi_j^{\ t}=g_{jh}-\eta_j\eta_h$,

(2.6) $\nabla_j\phi_{rs}\nabla_i\phi^{rs}=R_{ji}-2(g_{ji}-\eta_j\eta_i)$,

(2.7) $R_{jhrs}\phi^{rs}=-2\phi_{jh}$.

§ 3. Proof of Theorem.

In this section, we shall prove the theorem stated in § 0.

Let M be a compact n-dimensional SK-space. Then a 3-form $\overline{\Phi}=(1/3!)\overline{\Phi}_{ijh}dx^i\wedge dx^j\wedge dx^h$ can be associated to the structure, where we put $\overline{\Phi}_{ijh}=\nabla_i\phi_{jh}$. As the 2-form ϕ is Killing, it follows that $d\phi=3\overline{\Phi}$, which means $d\overline{\Phi}=0$.

Now, let us prove that the form $\overline{\Phi}$ is a conformal Killing 3-form. As we take account of the integral formula (1.5), we calculate $\overline{\Phi}^{ijh}\nabla^r\nabla_r\overline{\Phi}_{ijh}$ and $\overline{\Phi}^{ijh}\nabla_i\nabla^r\overline{\Phi}_{rjh}$. Making use of (2.3), (2.4) and $\overline{\Phi}^{ijh}\overline{\Phi}^{ab}{}_{h}R_{ijab}=-2\overline{\Phi}_{ijh}\overline{\Phi}^{ijh}$ obtained by (2.7), we have

(3.1) $\overline{\Phi}^{ijh}\nabla^r\nabla_r\overline{\Phi}_{ijh}=3(\nabla_iR_{jh}-\overline{\Phi}_{ijh})\overline{\Phi}^{ijh}$,

(3.2) $\overline{\Phi}^{ijh}\nabla_i\nabla^r\overline{\Phi}_{rjh}=\overline{\Phi}^{ijh}(\nabla_iR_{js}\phi_h^{\ s}-R_{ir}\overline{\Phi}_{jh}^{\ \ r}+\overline{\Phi}_{ijh})$.

Therefore the integral formula (1.7) reduces to

(3.3)
$$\int_M[(1/n-2)\overline{\Phi}^{ijh}\{4(n-3)\nabla_iR_{js}\phi_h^{\ s}-4(n-1)\overline{\Phi}_{ijh}+4R_i^{\ r}\overline{\Phi}_{rjh}\}+(1/2)A_{ijkh}A^{ijkh}]\,dV=0.$$

On the other hand, we have

$$(3.4)\qquad 0=\int_M \nabla_i (R_{js}\phi_h{}^s \bar{\phi}^{ijh})dV$$
$$=\int_M \nabla_i R_{js}\phi_h{}^s \bar{\phi}^{ijh} dV+\int_M (R_{js}\bar{\phi}_{ih}{}^s \bar{\phi}^{ijh}+R_{js}\phi_h{}^s \nabla_i \bar{\phi}^{ijh})dV.$$

Consequently, by virtue of (2.5), (2.6) and (3.4), the equation (3.3) becomes

$$\int_M [(4/n-2)T_{ij}T^{ij}+(4/n(n-2))(R-n(n-1))^2$$
$$+(1/2)A_{ijkh}A^{ijkh}]dV=0,$$

where we put $T_{ij}=R_{ij}-(R/n)g_{ij}$.

Hence it follows that M is an Einstein space and $\bar{\phi}$ is a conformal Killing 3-form. Since $\bar{\phi}$ is closed, from (1.4) and (2.4) we find

$$\nabla_i \nabla_j \phi_{lh}=-(g_{ij}\phi_{lh}+g_{il}\phi_{hj}+g_{ih}\phi_{jl}),$$

which means that 2-form ϕ is a special Killing form with constant 1. This completes the proof of Theorem.

Science University of Tokyo
Tokyo, Japan.

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