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PERFECT ONTO PROJECTIVITY IN SOME CATEGORIES OF HAUSDORFF SPACES

By Sung Sa Hong

1. Introduction.

B. Banaschewski has shown in [1] that perfect onto projectivity in a full subcategory of the category **Haus** of Hausdorff spaces and continuous maps which is left-fitting with respect to essential perfect onto maps, is properly behaved. In this paper, we show that for any extensive subcategory **C*** of the category **Haus*** of Hausdorff spaces and continuous semi-open maps, the full subcategory **C** of the category **Haus** determined by objects of **C*** is left-fitting with respect to essential perfect onto maps. Hence perfect onto projectivity in the category **C** is properly behaved. It is known that for any infinite cardinal k, the subcategory of almost k-compact spaces is extensive in the category **Haus***. Z. Frolik has shown in [3] that the class of regular almost realcompact spaces is left-fitting with respect to perfect onto maps. Using the above result, the class of almost k-compact spaces is left-fitting with respect to irreducible perfect onto maps. In particular, so are the class of H-closed spaces and the class of almost real compact spaces. We introduce the concept of almost k-boundedness for an infinite cardinal k greater than $\aleph_0$. Likewise compact spaces are exactly pseudocompact realcompact spaces (see [4]), a Hausdorff space is H-closed iff it is almost k-compact almost k-bounded. Consequently, a completely regular space is compact iff it is almost k-compact (or k-compact) almost k-bounded. Our terminology follows mainly [1]. In particular, Γ, $\mathscr{I}$ and $\mathscr{C}$ denote closure operator, interior operator and complement respectively.

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2. Perfect onto projectivity.

2.1 DEFINITION, Let **K** be a category and **P** a class of morphisms in **K**. An object A of **K** is said to be **P**-*projective* if for any $g:A\to B$ in **K** and for any $f:C\to B$ in **P**, there is a morphism $h:A\to C$ in **K** with $g=fh$. A morphism f in **P** is said to be *essential* if $fg\in$ **P** implies $g\in$**P**. A morphism $f:A\to B$ in **K** is said to be a **P**-*projective* cover of B if A is **P**-projective and f is essential.

The following definition is due to B. Banaschewski [1].

2.2 DEFINITION. Let **K** be a category and **P** a class of morphisms in **K**. The **P**-projectivity is said to *behave properly* if the following three conditions are fulfilled:

1. The following are equivalent for an object A:
   a) A is **P**-projective.
   b) Any morphism $f:B\rightarrow A$ in **P** has a right inverse.
   c) Any essential morphism $f:B\rightarrow A$ is an isomorphism.
2. Any object in **K** has an essentially unique **P**-projective cover.
3. The following are equivalent for a morphism $f:B\rightarrow A$ in **P**:
   a) f is a **P**-projective cover.
   b) f is an essential morphism and for any g, if fg is an essential morphism then g is an isomorphism.
   c) B is **P**-projective, and if $f=hg$ with morphisms g and h in **P** where h has **P**-projective domain then g is an isomorphism.

In what follows, we deal with subcategories of the category **Haus** and **P** will be the class of perfect onto morphisms in the category. Instead of **P**-projectivity we will call it p.o. projectivity.

2.3 DEFINITION. If X and Y are topological spaces, then a map $f:X\rightarrow Y$ is said to be *semi-open* if the image under f of each non-empty open set of X has non-empty interior in Y.

H. Herrlich and G.E. Strecker have shown [7] that the full subcategory of **Haus*** determined by H-closed spaces is epi-reflective in **Haus*** via the Katětov extension.

2.4 DEFINITION. A reflective subcategory **C*** of **Haus*** is said to be *extensive* in **Haus*** if every **C***-reflection map is a dense embedding.

2.5 DEFINITION. An onto map $f:X\rightarrow Y$ in **Haus** is said to be *irreducible* if for any closed $A\subset X$, $f(A)=Y$ implies $A=X$.

It is known [1] that a perfect onto map in **Haus** is essential iff it is irreducible.

2.6 LEMMA. *Every irreducible closed map is semi-open.*

PROOF. Let $f:X\to Y$ be an irreducible closed map and let U be a non-empty open set of X. Since f is irreducible closed, $\mathscr{C}f(\mathscr{C}U)$ is non-empty open and $f(U)$ contains $\mathscr{C}f(\mathscr{C}U)$. Hence $f(U)$ has a non-empty interior.

2.7 THEOREM. *Every extensive subcategory* $\mathbf{C}^*$ *of the category* **Haus*** *is left-fitting with respect to essential perfect onto maps in* **Haus**, *i.e. for any essential perfect onto map* $f:X\to Y$, *X belongs to* $\mathbf{C}^*$ *whenever Y belongs to* $\mathbf{C}^*$.

PROOF. Let $f:X\to Y$ be an essential perfect onto map and Y an object of $\mathbf{C}^*$. By Lemma 2.6, f is a morphism of the category **Haus***. Let $r_X:X\to rX$ be the $\mathbf{C}^*$-reflection of X. Then there is a unique morphism $\bar{f}:rX\to Y$ in $\mathbf{C}^*$ with $\bar{f}r_X=f$. By Lemma 7, in [1], $\bar{f}(rX-r_X(X))\subset Y-f(X)=\phi$ and so $rX=r_X(X)$. Hence r_X is an onto homeomorphism, which implies X belongs to $\mathbf{C}^*$.

REMARK. Since every continuous semi-open map is a p-map (see [5]), Theorem 2.7 still holds for an extensive subcategory of the category **pHaus** of Hausdorff spaces and p-maps.

2.8 COROLLARY. *The class of H-closed spaces is left-fitting with respect to irreducible perfect onto maps.*

Using Proposition 3 and Corollary 3 of Proposition 4 in [1], the following is immediate from Theorem 2.7.

2.9 THEOREM. *Let* $\mathbf{C}^*$ *be an extensive subcategory of the category* **Haus*** *and* **C** *be the full subcategory of the category* **Haus** *determined by objects of* $\mathbf{C}^*$. *Then p.o. projectives in* **C** *are exactly extremally disconnected spaces belonging to* **C** *and p.o. projectivity in* **C** *is properly behaved.*

3. Almost k-compact spaces.

Z. Frolik has introduced in [3] almost realcompact spaces and R.N. Bhaumik and D.N. Misra have generalized it in [2] to almost k-compact spaces for any infinite cardinal k.

3.1 DEFINITION. Let k be an infinite cardinal. A Hausdorff spaces X is said to be *almost k-compact* if a maximal open filter $\mathscr{U}$ on X for which $\{\Gamma U \mid U\in\mathscr{U}\}$ has the k-intersection property is convergent.

We note that almost $\aleph_0(\aleph_1)$-compact spaces are precisely H-closed (almost realcompact) spaces.

3.2 DEFINITION. Let k be an infinite cardinal number greater than $\aleph_0$.

A Hausdorff space X is said to be *almost k-bounded* if every open covering of X with cardinal less than k has a finite proximate subcovering, i.e. a finite subfamily whose union is dense in X.

We note that a completely regular space is pseudocompact iff it is almost $\aleph_1$-bounded.

The following theorem is immediate from the fact that every open filter in an almost k-bounded space has the k-intersection property.

3.3 THEOREM. *Let k be an infinite cardinal greater than $\aleph_0$. Then a Hausdorff space is H-closed iff it is almost k-compact almost k-bounded.*

3.4 DEFINITION. Let α be a collection of open coverings of a space X. An α-Cauchy family is an open filter base $\mathscr{B}$ on X such that for every $\mathfrak{A}$ in α, there exists an $A \in \mathfrak{A}$ and a $B \in \mathscr{B}$ with $B \subset A$. The space X is said to be *α-complete* if every α-Cauchy family has at least one cluster point.

The following definition is due to H. Herrlich [6].

3.5 DEFINITION. A completely regular space is said to be *k-compact* if every z-ultrafilter with the k-intersection property is fixed.

3.6 THEOREM. *Let X be a completely regular space and let $\mathscr{L}_k$ be the family of cozero set coverings of X with cardinal less than k. Then X is k-compact iff it is $\mathscr{L}_k$-complete.*

PROOF. It is obvious that the space X is compact $(=\aleph_0$-compact) iff X is $\mathscr{L}_{\aleph_0}$-complete. Hence we may assume that k is greater than $\aleph_0$. Let $\mathscr{U}$ be a $\mathscr{L}_k$-Cauchy family in a k-compact space X. Suppose that $\mathscr{U}$ has no cluster point. Since $\mathscr{U}$ is a filter base on its Stone-Čech compactification βX. $\mathscr{U}$ has a cluster point in βX, say $p \in \beta X - X$. Since X is k-compact, it is k-closed in βX (see 1.8. in [8]); there is a family $(U_i)_{i\in I}$ of open neighborhoods of p in βX such that $\bigcap U_i \cap X = \phi$ and $|I| < k$. Using the fact that the zero-set neighborhoods of p in βX form a fundamental system of neighborhoods of p, there exists a family $(Z_i)_{i\in I}$ of zero-sets of βX such that $p \in \mathscr{I} Z_i \subset Z_i \subset U_i$ for each $i \in I$. Thus $(\bigcap Z_i) \cap X = \phi$. Obviously, $\{\mathscr{C}_X(Z_i \cap X) \mid i \in I\}$ is a member of $\mathscr{L}_k$ and so there is a $U \in \mathscr{U}$ and an $i \in I$ with $U \subset \mathscr{C}_X(Z_i \cap X)$. Hence $U \cap Z_i = \phi$ which is a contradiction to that p belongs to the closure of U in βX.

Conversely, let $\mathscr{F}$ be a z-ultrafilter on a $\mathscr{L}_k$-complete space X with the k-

intersection property. Consider $\mathscr{U}=\{U \mid U$ is open and contains a member of $\mathscr{F}\}$. Then it is easy to show that $\mathscr{U}$ is $\mathscr{L}_k$-Cauchy. Hence $\mathscr{U}$ has a cluster point. By the complete regularity of X, we have $\bigcap\{Z \mid Z \in \mathscr{F}\}=\bigcap\{\Gamma U \mid U \in \mathscr{U}\} \neq \phi$. Thus X is k-compact.

Since a Hausdorff space X is almost k-compact iff X is α_k-complete, where α_k is the family of all open coverings of X with cardinal less than k (see [2]), the following is immediate from Theorem 3.6.

3.7 COROLLARY. *Every k-compact space is almost k-compact.*

3.8 COROLLARY. *For a completely regular space X, the following are equivalent:*

1) *X is compact.*
2) *X is k-compact almost k-bounded.*
3) *X is almost k-compact almost k-bounded.*

PROOF. It is immediate from the fact that every regular H-closed space is compact.

C-T. Liu and G. E. Strecker have shown that the full subcategory of **Haus*** determined by almost realcompact spaces is extensive in **Haus*** (see [10]). By the same argument, we have:

3.9 THEOREM. *The full subcategory* $\mathbf{A}_k{}^*$ *of* **Haus*** *determined by almost k-compact spaces is extensive in* **Haus***.

We note that the $\mathbf{A}_k{}^*$-reflection of a Hausdorff space X is given by $X \cup \{\mathscr{U} \mid \mathscr{U}$ is a non-convergent maximal open filter such that $\{\Gamma U \mid U \in \mathscr{U}\}$ has the k-intersection property$\}$ with the relative topology of the Katětov extension κX of X and the natural embedding.

3.10 COROLLARY. *The class of almost k-compact spaces is left-fitting with respect to irreducible p.o. maps. P.o. projectivity in the category of almost k-compact spaces and continuous maps is properly behaved. An almost k-compact space is p.o. projective in the category iff it is extremally disconnected.*

3.11 LEMMA. *Let* ω_α *be the first ordinal of cardinal* $\aleph_\alpha$. *Let* $W(\omega_{\alpha+1})$ *be the space of ordinal less than* $\omega_{\alpha+1}$ *endowed with the order topology. Then a family with cardinal less than* $\aleph_{\alpha+1}$, *formed of closed cofinal subsets of* $W(\omega_{\alpha+1})$ *has a non-empty intersection.*

PROOF. Let $(F_i)_{i \in I}$ be such a family. Choose a relation $<$ which well-orders I.

Let Σ be an ordered set $I\times N$ with the order relation $\leqslant$, where $(i, n)\leqslant(j, m)$ iff $n<m$ or $i\leqslant j$ if $n=m$ and N is the set of natural numbers with the usual order relation. Then by the induction, one can construct a subset $A=\{\lambda_{i,n}|(i,n)\in\Sigma\}$ of $W(\omega_{\alpha+1})$ such that $\lambda_{i,n}\in F_i$ and $(i,n)\leqslant(j, m)$ implies $\lambda_{i,n}\leqslant\lambda_{j,m}$. Since the cardinal of A is less than $\aleph_{\alpha+1}$, it is bounded in $W(\omega_{\alpha+1})$; $\sup A$ exists. Furthermore, one can easily show that for each $i\in I$, $\sup A=\sup\{\lambda_{i,n}|n\in N\}$. Thus we can conclude that $\sup A$ belongs to $\cap F_i$.

3.12 COROLLARY. *The space $W(\omega_{\alpha+1})$ is almost $\aleph_{\alpha+2}$-compact but not almost $\aleph_{\alpha+1}$-compact.*

PROOF. It is known [6] that the space $W(\omega_{\alpha+1})$ is $\aleph_{\alpha+2}$-compact. Hence it is almost $\aleph_{\alpha+2}$-compact by Corollary 3.7. On the other hand, let $\mathscr{U}$ be a maximal open filter containing $\{T(\sigma+1)\} \ \sigma<\omega_{\alpha+1}\}$, where $T(\sigma)=\{\tau|\sigma\leqslant\tau<\omega_{\alpha+1}\}$. Then it is clear that $\mathscr{U}$ is not convergent. By Lemma 3.11, $\{\Gamma U|U\in\mathscr{U}\}$ has the $\aleph_{\alpha+1}$-intersection property, for every member of a non-convergent maximal open filter on $W(\omega_{\alpha+1})$ is cofinal.

REMARK. Van der Slot has asserted in [12] that the space $W(\omega_{\alpha+1})$is $\aleph_{\alpha+2}$-ultracompact but not $\aleph_{\alpha+1}$-ultracompact. But his proof was incorrect. However, using Lemma 3.11, one can prove his assertion.

3.13 THEOREM. *For any limit cardinal k, there exists an almost k-compact space which is not almost t-compact for every infinite cardinal t less than k.*

PROOF. Let X be the space $\Pi W(\omega_{\alpha+1})$ $(\aleph_{\alpha+1}<k)$ with the product topology. Since X is k-compact (see [6]), X is almost k-compact. By the same argument as Theorem 5 in [3], every closed subspace of an almost t-compact regular space is also almost t-compact. Hence we can conclude that X is not almost t-compact for $t<k$. Indeed, suppose X is almost $\aleph_\alpha$-compact for some $\aleph_\alpha<k$. Then the closed subspace of X which is homeomorphic with the space $W(\omega_{\alpha+1})$ is also almost $\aleph_\alpha$-compact, which is a contradiction.

Carleton University
Ottawa, Ontario
Canada

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