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A NOTE ON RECURRENCE RELATIONS FOR A SET OF POLYNOMIALS SUGGESTED BY LAGUERRE POLYNOMIALS

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1. In his paper Konhauser [1] examined the biorthogonality of the pair $\{Z_n^c(x;k)\}$, $\{Y_n^c(x;k)\}$ with respect to the weight function $e^{-x}x^c$ over the interval $(0,\infty)$ and obtained among other things a recurrence relation for the polynomials $Z_n^c(x;k)$ connecting polynomials corresponding to c and $c+k$ by direct calculations. Prabhakar [2] established a pure recurrence relation for the polynomial set $Z_n^c(x;k)$ by using Schläfli's contour integral.

The purpose of this note is to derive a simple method by means of which we can obtain short proofs of results of Konhauser [1,(8)] and Prabhakar [2,(2.6)].

2. It is known [2] that the generating function for the polynomial set $Z_n^c(x;k)$ is given by

$$e^t\phi(k,c+1\,;\,-x^kt)=\sum_{n=0}^{\infty}\frac{Z_n^c(x;k)\ t^n}{\Gamma(kn+c+1)},\tag{2.1}$$

where $\phi(a,b;z)$ is the Bessel-Maitland function [4,(1.3)].

Let

$$G=e^t\phi(k,c+1;-x^kt).\tag{2.2}$$

Differentiating partially (2.2) with respect to x and t and using [4, (2.2)], we have

$$\frac{\partial G}{\partial x}=-kx^{k-1}\ te^t\ \phi(k,c+k+1;-x^kt)\tag{2.3}$$

and

$$\frac{\partial G}{\partial t}=e^t\phi(k,c+1;\ -x^kt)-x^ke^t\phi(k,c+k+1;\ -x^kt).\tag{2.4}$$

From (2.1), (2.2) and (2.3), we get

$$\sum_{n=0}^{\infty}\frac{\frac{d}{dx}Z_n^c(x;\ k)}{\Gamma(kn+c+1)}t^n=-kx^{k-1}\sum_{n=0}^{\infty}\frac{Z_n^{c+k}(x;\ k)}{\Gamma(kn+c+k+1)}t^{n+1}.\tag{2.5}$$

Equating coefficients of t^n on both sides of (2.5), we have

$$\frac{d}{dx}Z_n^c(x;k)=-kx^{k-1}\ Z_{n-1}^{c+k}(x;\ k),$$

also obtained by Konhauser [1, (8)] by direct calculations.

From (2.1), (2.2) and (2.4), we derive

$$(2.6)\qquad \sum_{n=0}^{\infty} \frac{n\, Z_n^c(x;\ k)}{\Gamma(kn+c+1)} t^{n-1} = \sum_{n=0}^{\infty} \frac{Z_n^c(x;\ k)}{\Gamma(kn+c+1)} t^n - x^k \sum_{n=0}^{\infty} \frac{Z_n^{c+k}(x;k)}{\Gamma(kn+c+k+1)} t^n .$$

Equating coefficients of t^n on both sides of (2.6) and after a slight adjustment, we get

$$(2.7)\qquad x^k\, Z_n^{c+k}(x;\ k) = (kn+c+1)_k\, Z_n^c(x;\ k) - (n+1)\, Z_{n+1}^c(x;\ k),$$

also established by Prabhakar [2, (2.6)] by a different method.

If we put $k=2$ in (2.7), we obtain [3, (5.39)].

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REFERENCES

[1] Konhauser, J. D. E., *Biorthogonal polynomials suggested by the Laguerre polynomials*, Pacific J. Math. 21(1967), 303—314.

[2] Prabhakar, T. R., *On a set of polynomials suggested by Laguerre polynomials*, Pacific J. Math. 35(1970), 213—219.

[3] Preiser, S., *An investigation of biorthogonal polynomials derivable from ordinary differential equations of the third order*, J. Math. Anal. Appl. 4(1962), 38—64.

[4] Wright, E. M., *On the coefficients of power series having exponential singularities*, J. London Math. Soc. 8(1933), 71—79.