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A NOTE ON CYCLICITY OF GROUPS

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It is well known (see [1] p. 133) that if $|G|=p_1\cdot p_2$; $p_1<p_2$ where p_i's $(i=1,2)$ are primes then group G is cyclic whenever $p_2\neq 1+kp_1$ for any nonzero integer k. We generalize this fact by proving the following theorem.

THEOREM. *Let G be a group of order* $p_1\cdot p_2\cdots p_n$ *where* p_i*'s are distinct ordered primes. If G satisfies the following* $\frac{n}{2}[2^n-(n+1)]$ *conditions*:

(1) $1+kp_j\neq p_{\sigma(1)}\cdot p_{\sigma(2)}\cdots p_{\sigma(i)}$; $j\neq\sigma(i), \forall\sigma(i), 2\leq i\leq n-1$

(2) $1+kp_j\neq p_{\sigma(1)}$; $\sigma(1)=j+1, j+2, \cdots, n,$

where σ *is a permutation on* $\{1,2,\cdots,n\}$, k *is any nonzero integer; then G is cyclic.*

PROOF. In fact, (1) and (2) enable us to prove the existence of normal subgroups $C_{p_1}, \cdots, C_{p_{n-1}}$ and $C_{p_n}(=K)$ say, of order $p_1, p_2, \cdots, p_{n-1}$ and p_n respectively. Orderedness of primes supplements with (1) and (2) that direct product $C_{p_1}\times\cdots\times C_{p_n}$ is cyclic. The proof follows by using induction on n. If $n=1$ there is nothing to prove. For $n=2$ the theorem is well-known (cf. [1] p. 133). Assume that $H=C_{p_1}\times C_{p_2}\times\cdots\times C_{p_{n-1}}$ is cyclic by induction hypothesis. Now we consider a group G of order $q\cdot p_n=(p_1\cdots p_{n-1})\,p_n$. The number of Sylow p_j-subgroups in G is $1+k\cdot p_j$ which divides the order of G. For this there are following possibilities:

(i) $1+k\cdot p_j=p_{\sigma(1)}$; $\sigma(1)\leq j$

(ii) $1+k\cdot p_j=p_{\sigma(1)}$; $\sigma(1)=j+1, j+2, \cdots, n$

(iii) $1+k\cdot p_j=p_{\sigma(1)}\cdot p_{\sigma(2)}\cdots p_{\sigma(i)}$; $j=\sigma(i)$ for some i but $j\neq\sigma(1), \cdots, \sigma(i-1)$

(iv) $1+k\cdot p_j=p_{\sigma(1)}\cdot p_{\sigma(2)}\cdots p_{\sigma(i)}$; $j\neq\sigma(i), \forall\sigma(i)$

(v) $1+k\cdot p_j=1$ for all $j=1,2,\cdots,n-1$

All possibilities are ruled out by (1) and (2) and only (v) remains which means that C_{p_j} is normal for $1\leq j\leq n-1$. Now it is easy to prove that there exists a unique Sylow p_n-subgroup K, because out of the following four possibilities only last one persists and others are ruled out;

(i)′ $1+k\cdot p_n=p_{\sigma(1)}$; $\sigma(1)\leq n$

(ii)′ $1+k\cdot p_n=p_{\sigma(1)}\cdot p_{\sigma(2)}\cdots p_{\sigma(i)}$; $\sigma(i)=n$ for some i

(iii)′ $1+k\cdot p_n=p_{\sigma(1)}\cdot p_{\sigma(2)}\cdots p_{\sigma(i)}$; $\sigma(i)\neq n, \forall\sigma(i)$

(iv)′ $1+k\cdot p_n=1$

Thus $H=C_{p_1}\times\cdots\times C_{p_{n-1}}$ and $K=C_{p_n}$ are normal in G and by the same arguements as we do in particular case when $n=2$ (see [1] p. 133), we can easily show that G is cyclic of order $q\cdot p_n=p_1\cdots p_n$. This completes the proof.

REMARK 1. If $|G|=p_1\cdot p_2\cdots p_n$ then the fact that number of conditions mentioned in (1) and (2) are $\frac{n}{2}[2^n-(n+1)]$, can be verified as follows:

(1) includes $n({}^{n-1}C_2+\cdots+{}^{n-1}C_{n-1})$ number of conditions, and

(2) includes $((n-1)+\cdots+(n-n))$ number of conditions.

Thus total number is $\frac{n}{2}[2^n-(n+1)]$.

REMARK. 2. The example for $n=2$ is well-known, namely $|G|=15$. A natural question arises for $n>2$. To this end we exemplify the case $n=3$. Consider the group G where $|G|=7\times 11\times 13$ with the following conditions.

(a) $1+7k\neq 11$ (b) $1+7k\neq 13$ (c) $1+11k\neq 13$

(d) $1+7k\neq 11\times 13$ (e) $1+11k\neq 7\times 13$ (f) $1+13k\neq 7\times 11$

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REFERENCE

[1] Baumslag B. and B. Chandler, *Group Theory* Schaum Outline Series, McGraw Hill Newyork 1968.