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LATTICE STRUCTURE OF GENERAL TOPOLOGICAL EXTENSIONS

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§ 0. Introduction.

In [7], the author has characterised the epireflection $\beta_E X$ of a space X in an epireflective full subcategory $\boldsymbol{E}$ of the category $\boldsymbol{T2}$ of all Hausdorff spaces, as the space of 'the largest imagive determining type' of nets in X modulo a natural equivalence, topologised in a natural way. In this paper we define E-extensions of E-regular spaces and study the lattice structure of the collection of all E-extensions under a natural order. They form an upper-complete semi-lattice. But further study is rather difficult in such a general set up. So we put some restrictions on the property $\boldsymbol{E}$ (we do not distinguish between 'properties' and 'full subcategories') and/or on the spaces for which E-extensions are sought. We introduce the notion of generating families $\boldsymbol{gE}$ or $\boldsymbol{E}$ and study them in detail. In particular, we obtain certain partial results to the problem of when one-point-E-extensions exist. We also give certain equivalent forms of being hereditarily $\boldsymbol{E}$ when $\boldsymbol{E}$ has a strongly hereditary pseudoconvergent determining type of nets.

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CONVENTION. $\boldsymbol{E}$ is a full subcategory of $\boldsymbol{T2}$ which is also assumed to be epireflective unless otherwise stated. All spaces considered are objects of $\boldsymbol{T2}$.

§ 1. Preliminaries.

1.1. DEFINITION. A space X is said to be *E-regular* if it is homeomorphic to a subspace of a product of spaces in $\boldsymbol{E}$.

1.2. DEFINITION. Let X be a space. If Y is in $\boldsymbol{E}$ such that X is homeomorphic to a dense subset of Y, we say that Y is an *E-extension* of X.

1.3. DEFINITION. Let $\boldsymbol{E}$ be a subcategory of $\boldsymbol{T2}$ not necessarily epireflective. Let $\boldsymbol{NE}$ associate to each space X, a class of nets $\boldsymbol{NE}(X)$ in X such that X has property $\boldsymbol{E}$ if and only if every net in $\boldsymbol{NE}(X)$ converges to some point in X.

Then NE is called a *determining type of nets* for E or that *E is determined by NE.*

REMARK. Determining types of nets have been extensively studied by the author in [7]. We quote the following results from [7].

1.4 RESULT. Let E be a full subcategory of $\boldsymbol{T2}$. There exists a determining type of nets NE for E if and only if the empty space as well as the singleton space has E.

1.5. DEFINITION. The type of nets N is called *imagive* if whenever $f: X \to Y$ is continuous, $f(N(X)) \subset N(Y)$, where $f(N(X)) = \{f \circ s \mid s \in N(X)\}$.

1.6 RESULT. A full subcategory E of $T2$ is epireflective if and only if there exists an imagive determining type of nets NE for E.

NOTE. $NE_i(X) = \{s \mid s$ is a net in X such that if Y is in E and $f: X \to Y$ any continuous map, then $f \circ s$ converges in $Y\}$, is 'the largest imagive determining type of nets' for an epireflective subcategory E of $\boldsymbol{T2}$.

1.7 REMARK. In [7], the author has proved that the epireflection $\beta_E X$ is the space of equivalence classes of $NE_i(X)$ topologised in a natural way.

§ 2. E-extensions of E-regular spaces and generating families for E.

2.1 DEFINITION. Let X be an E-regular space and let $aX, a'X$ be E-extensions of X. Then we say that $aX \leq a'X$ if there exists a continuous map from $a'X$ into aX, with identity on X.

2.2 THEOREM. *The collection of all E-extensions of an E-regular space X forms a complete upper semi-lattice under the partial order generated by the pre-order defined above.*

PROOF. The proof is analogous to that in [1] for the corresponding theorem for compactifications.

Let $\{a_i X\}_{i \in J}$ be a family of E-extensions of X. To show that there exists an E-extension aX of X such that $aX = \bigvee_i a_i X$. Let $f: X \to \Pi_i a_i X$ be defined as $f(x) = (a_i(x))_{i \in J}$ where $a_i(x)$ denotes the image of x in $a_i X$ under the homeomorphic embedding. It can be easily checked that f is a homeomorphism (cf. [5] Theorem 2.1). Now $\mathrm{cl}.f(X) = aX$ is an E-extension, being a closed subspace of a product of spaces in E. Further $aX \geq a_i X$ for each i in J since the projection from aX

onto a_iX is continuous and is identity on X. Also if $a'X$ is an E-extension of X such that $a'X \geq a_iX$ for every i, then $a'X \geq aX$ since the product continuous function $\Pi_i f_i$ suffices where $f_i : a'X \to a_iX$. Thus $aX = \vee_i a_iX$.

NOTE. $\beta_E X$, the epireflection of X in E is the largest E-extension in this order. In particular, when X has E, then X is its own largest E-extension.

REMARK. To get more information about the semilattice in this most general set up is rather difficult. So we introduce certain restrictions on the property E and/or on the spaces for which E-extensions are sought. We introduce the notion of a generating family for the property E.

2.3 DEFINITION. A collection of spaces $(Y_i)_{i \in J}$ is called a ***generating family*** for E if the following happens: a space X has E if and only if X is homeomorphic to a closed subspace of a product of spaces $(Y_i)_{i \in K}$, $K \subset J$. If there exists a finite (respectively singleton) generating family for E then E is said to be finitely (respectively singly) generated.

EXAMPLE. (i) If I is the closed unit interval of reals, then (I) is a generating family for compactness.

(ii) If J is the open unit interval of reals, then (J) is a generating family for realcompactness.

(iii) The discrete dyad D forms a generating family (D) of being zero dimensional and compact.

NOTE. It is not hard to see that a property E is finitely generated if and only if singly generated. Some singly generated epireflective full subcategories of $\boldsymbol{T2}$ are studied by S. Mrowka in [5].

2.4 RESULT. The generating families for a property E form an upper complete semilattice under inclusion, This semilattice possesses minimal elements if and only if E is singly generated. It is not necessarily a lattice even when E is singly generated.

The first two statements can be proved easily. To justify the last statement, consider E=realcompactness. If $J=(0,1)$, then it is known that (J) is a generating family for E. It can be fairly easily proved that if $K=[0,1)$, then (K) is also a generating family for E. $(J) \cap (K) = \phi$ is clearly not a generating family.

CONVENTION. In the rest of the paper gE stands for a typical generating family for E.

2.5 REMARK. Let E be a closed hereditary productive property. If we define $NE(X)=\{s \mid \text{for any } Y \text{ in } gE \text{ and for any continuous function } f: X\to Y,\ f\circ s \text{ is convergent in } Y\}$, then it can be seen that NE is an imagive determining type of nets for E. Hence or independently it is seen that NE is precisely NE_i, the largest imagive one.

2.6 DEFINITION. A subset A of X is said to be *gE-embedded* in X if every continuous function on A to any space Y in gE extends continuously to X.

2.7 RESULT. If a closed subset A is gE-embedded in X, and $\beta_E X$ is the largest E-extension of X, then $\mathrm{cl}_{\beta_E X} A=\beta_E A$.

The proof is easy and omitted.

NOTE. The converse of Result 2.7 is not true. Example : E=realcompactness, $gE=(J)$ where J is the open unit interval of reals and X is a non-normal realcompact space.

2.8 DEFINITION. An E-regular space X is said to be *gE-normal* if every closed subset of X is gE-embedded in X.

NOTE. gE-normal$\Rightarrow$normal.

2.9 REMARK. Herrlich, H. [3] has defined a space to be E-normal if every disjoint pair of closed subsets in it is E-separated, i.e, if whenever A and B are disjoint closed subsets of X, there exists an E-space Y and a continuous function $f: X\to Y$ such that $\mathrm{cl} f(A)\cap \mathrm{cl} f(B)=\phi$. It can be easily seen that gE-normal$\Rightarrow$ E-normal for any generating family gE of E. Notice that any E-space is E-normal; but not necessarily gE normal. For example, $R_l x R_l$ where R_l is the set of reals with lower limit topology is realcompact and (realcompact)-normal in the sense of Herlich. But it is not gE-normal for any gE since it is not normal. Notice that in particular (R)-normal if and only if normal. (cf. [2] 3D1 (48)). Thus again, a converse of result 2.7 is not true; however a partial converse can be given as follows:

2.10 RESULT. If X or $\beta_E X$ is gE normal, then whenever $\mathrm{cl}_{\beta_E X} Y=\beta_E Y$, we have Y is gE-embedded in X.

The proof is easy and omitted.

NOTE. Now we come back to the consideration of the collection of E-extensions, and in particular, to the problem of existence of one-point-E-extensions.

2.11 DEFINITION. A regular space X is called *locally* E if each point of X has a basis of E-neighbourhoods.

2.12 THEOREM. *Let X be E-regular and $\beta_E X$ regular. If X is open in $\beta_E X$, then X is locally E. On the other hand, if X is locally E and gE-normal for some generating family gE, then X is open in $\beta_E X$.*

PROOF. Suppose X is open in $\beta_E X$; since $\beta_E X$ is regular, it follows that X is locally E.

Conversely, suppose X is locally E. If X has E, then trivial, since, $\beta_E X=X$ in that case. If X does not have E, we consider $\beta_E X$ as the space of equivalence classes of $NE_i(X)$ as in [7]. Suppose, if possible, $s: D\to\beta_E X$ is a net in $\beta_E X-X$ converging to a point p in X. Then for each j in D, $s(j)$ is a net in $NE_i(X)$ converging to a point $s(j)$ in $\beta_E X$. Consider the product net P in X corresponding to s. This product net P converges to p. Take a neighbourhood V of p in $\beta_E X$ such that $V\cap X$ is closed in X and has E. The net P is eventually in $V\cap X$ (say) after $(m_0, f_0)\in D\times \prod_{i\in D} E_i$. Consider $s(m)$ where $m>m_0$. It can be easily seen that this net which is a member of $NE_i(X)$ is eventually in $V\cap X$. But since X is gE-normal, by remark 2.5, it follows that $s(m)\in NE_i(V\cap X)$. Now $V\cap X$ has E so that $s(m)$ converges inside $V\cap X$, which is a contradiction. Hence any net in $\beta_E X-X$ converges in $\beta_E X-X$, if at all it converges. Hence X is open in $\beta_E X$.

NOTE. X is open $\beta_E X$ does not imply that X is gE normal for any gE. For example, suppose X is realcompact and non-normal. Then trivially X is open in $\nu X=X$. But since X is not normal, it is not gE-normal for any gE where $E=$ realcompactness.

QUESTION. If X is E-regular and locally E, $\beta_E X$ is regular and if either X or $\beta_E X$ is gE-normal, for some generator gE of E, then does it follow that X is open in $\beta_E X$?

2.13 DEFINITION. A property E is *collapsible* if for every space X which is a proper open subset of $\beta_E X$, the identification of $\beta_E X-X$ to a point has E.

2.14 RESULT. Let E be collapsible. Let X be E-regular and gE-normal. Suppose X does not have E. Further let $\beta_E X$ be regular. Then X has a one-point-E-

extension, if and only if X is locally $\boldsymbol{E}$.

2.15 REMARK. If X has $\boldsymbol{E}$ and has a one-point-$\boldsymbol{E}$-extension, then X is not $\boldsymbol{E}$-closed. (we call a space $\boldsymbol{E}$-closed, if it is closed in every space with $\boldsymbol{E}$ containing it). On the other hand, if X is not $\boldsymbol{E}$-closed, then a sufficient condition that a one-point-$\boldsymbol{E}$-extension exists is that the union of a compact subset and any subset having $\boldsymbol{E}$ has again $\boldsymbol{E}$. We will describe certain such situations in then next section.

§ 3. Pseudoconvergent determining types of nets and Hereditarily E spaces.

3.1 DEFINITION. A determining type of nets $\boldsymbol{NE}$ is called *strongly hereditary* if whenever $A \subset X$, then $\boldsymbol{NE}(A) \subset \boldsymbol{NE}(X)$ and furthermore if s is a net in A such that $s \in \boldsymbol{NE}(X)$ then $s \in \boldsymbol{NE}(A)$.

3.2 DEFINITION. A type of nets N is called *pseudoconvergent* if every net in $N(X)$ is pseudoconvergent for any X.

CONVENTION. In this section we consider epireflective subcategories $\boldsymbol{E}$ of $\boldsymbol{T2}$ such that $\boldsymbol{E}$ has a strongly hereditary pseudoconvergent determining type of nets $\boldsymbol{NE}$.

EXAMPLES. (for details see [6])

(i) $\boldsymbol{E}$=compactness; $\boldsymbol{NE}$= {Universal nets} or= {weakly open universal nets}

(ii) $\boldsymbol{E}=\alpha'$-spaces; $\boldsymbol{NE}$= {σ-directed weakly open-universal nets}.

(iii) $\boldsymbol{E}=\alpha''$-spaces; $\boldsymbol{NE}$= {σ-directed strongly closed-universal nets}.

3.3 THEOREM. *If $\boldsymbol{NE}$ is an imagive strongly hereditary pseudoconvergent determining type of nets for $\boldsymbol{E}$ and if $\beta_E X$ is regular for an $\boldsymbol{E}$-regular space X, then X is open in $\beta_E X$ if and only if X is locally $\boldsymbol{E}$.*

Proof easy and omitted.

3.4 THEOREM. *If $X=Y \cup Z$ where Y has $\boldsymbol{E}$ and Z is compact, then X has $\boldsymbol{E}$.*

PROOF. Let $s \in \boldsymbol{NE}(X)$. If s is frequently in Z, then it has a convergent subnet and since s is pseudoconvergent, it follows that s converges. If s is eventually in $cZ \subset Y$, then $s \in \boldsymbol{NE}(Y)$ since $\boldsymbol{NE}$ is strongly hereditary. Now since Y has $\boldsymbol{E}$, s is convergent. Hence X has $\boldsymbol{E}$.

3.5. NOTE. See Remark 2.15.

3.6 THEOREM. *The following are equivalent on a space Y:*

(a) *Y is hereditarily $\boldsymbol{E}$, i.e., every subspace of Y has $\boldsymbol{E}$.*

(b) *For each space X, if there exists a map $f: X \to Y$ such that $f^{-1}(y)$ is compact* *for each y in Y, then Y has E.*

(c) *Every space of which Y is a one-one continuous image, has E.*

(d) *For each point y in Y, $Y-\{y\}$ has E.*

PROOF. Use Theorem 3.4 and proceed along the same lines as those of Theorem 8.17 in [2] (122).

To show (c) implies (a), the technique employed is the same, by noticing the following: Given a subspace $Z \subset Y$, by enlarging the topology of Y making Z and $Y-Z$ open, the new space obtained is E-regular.

3.7 COROLLARY. *If $f: X \rightarrow Y$ is one-one and continuous and if Y is hereditarily E, then X is hereditarily E.*

3.8 NOTE. If every subspace of Y has E, and if card $Y=m$, then every discrete space of cardinality $\leq m$ has E.

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