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DIMENSION OF AUTOMORPHIC FORMS OF EXTENDED KLEINIAN GROUP

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Introduction.

In this paper we prove several identities about the autom rphic forms of extended group, also we observe that the Bers' Area theorem and Ahlfors' finiteness theorem of the Kleinian group extends naturally to the extended Kleinian group.

We list here some notations which we need in the sequel. E denotes a non-elementary extended group, and G is the maximal Kleinian group contained in E. $A_q^p(D,E)$ denotes the Banach space of holomorphic P-integrable automorphic form f weight $-2q$. By D we denote an invariant union of region of discontinuity of E and by D/G (or D/E) we denote the corresponding orbit space of the group. All the other terminologies undefined in this paper one can see in Kim [13] or Kra [9].

Let G be a (non-elementary) Kleinian group and let D be an invariant union of components of the region of discontinuity of G. Then we have the following lemma proved by Ahlfors (1).

LEMMA 1. *Let $D/G=S-p$ where S is a Riemann surface and $p\in S$. Assume further that there is a punctured neighborhood V of p on S such that the natural projection π_0: $D\to D/G$ is unramified over V. Then there exists a parabolic element $B\in G$ with fixed point q contained in the limit set of G, and there is a Möbius transformation A with the following properties.*

(1) *$A(\infty)=q$ and $A^{-1}BA$ is the translation $z\to z+1$,*

(2) *$A^{-1}(D)$ contains the half plane $U_e=\{z\in C: \mathrm{Im}(z)>e\}$,*

(3) *two points z_1 and z_2 over $A(U_e)$ are equivalent under G if and only if $z_2=B^n(z_1)$ for some integer n, and*

(4) *the image $A(U_e)$ under π_0 is a deleted neighborhood of P, homeomorphic to a punctured disc.*

We shall call p a *parabolic puncture*, U_e a half plane *belonging to* p and $\{z$: Im $(z)>e$, $1>\text{Real}(z)\geq 0\}$ a *cusped region* belonging to p.

We are going to establish a similar lemma for a (non-elementary) extended group E, and we let D be an invariant union of components of the region of discontinuity of the group E.

LEMMA 2. *Let $D/G=S-p$ where S is a Riemann surface (not necessarily connected) and p be a parabolic puncture. Let U_e be the half plane corresponding to p. Let π_0 and π be natural projections of D onto D/G and D/E respectively.*

Then (U_e) has one of the following properties. Either

(5) *$\pi(U_e)$ is conformally equivalent to $\pi_0(U_e)$ and two points z_1 and z_2 of U_e are equivalent under E if and only if $z_2=A^n(z_1)$ where A is the parabolic element corresponding to p, and maps U_e into U_e,*

or

(6) *$\pi(U_e)$ is a punctured half disk homeomorphic to $\{z$: $\text{Im}(z)\geq 0$ and $z\neq 0$, $|z|<1\}$. Two elements z_1 and z_2 of U_e are equivalent under E if and only if $z_2=k(z_1)$ for some element k in the subgroup generated by A and g, where g is an antianalytic element which fixes a circle. Taking the conjugation by a Möbius transformation we can normalize so that $A(z)=z+1$ and $g(z)=-\bar{z}$.*

PROOF. Let us assume that A is the parabolic element and has form $A(z)=z+1$, and $U_e=\{z$: Im $(z)>e\}$ be the half plane corresponding to p. Consider the following maps;

$$U_e/\{A\}\xrightarrow{\ i\ }D/G\xrightarrow{\ \pi_1\ }D/E$$

where i is the inclusion map and π_1 is the natural projection. Note that π_1 is a two-to-one map. Let $E=G\cup H$ and suppose there is no element in H which fixes ∞, then $g\in H$ has the following form:

$$g(z)=\frac{a\bar{z}+b}{c\bar{z}+d},\ ad-bc=1,\ c\neq 0 \text{ and } r(\infty)=\frac{a}{c}.$$

Then gAg^{-1} is a parabolic element fixing $\frac{a}{c}$ and $\frac{a}{c}$ is inequivalent to under group G. We observe that $\frac{a}{c}$ corresponds to a new parabolic puncture and it has $g(U_e)$ as its half plane. Take $x\in U_e$ and $g(x)\in g(U_e)$ then $\pi_0(x)$ and $\pi_0(g(x))$ are different points in D/G. Since π_1 is a two-to-one map, we know that $\pi_1\circ i$ is a one-to-one mapping on $U_e/\{A\}$, and $\pi_1(U_e)$ satisfies (5). New suppose there is an element in H which fixes ∞, then it can be written as $g(z)=re^{i\theta}\bar{z}+b$. Since

$g^2(U_c)\cap U_c$ is not empty, we have $g^2(z)=A^n(z)=z+n$. By a simple calculation we know that $r=1$ and $e^{i\theta}\bar{b}+b$ is an integer. Furthermore, gAg^{-1} fixes ∞, hence $ne^{i\theta}$ is an integer and $\theta=0$ or π. We conclude g has one of the following forms:

(8) $\qquad g(z)=\bar{z}+b$, or

(9) $\qquad g(z)=-\bar{z}+b$ where b is a real number.

If g has form (8) then

$$U_k=\{z\colon \operatorname{Im}(z)<-e-|b|=k\}$$

is also contained in D, and U_k is a half plane corresponding to a parabolic puncture of D/G. If U_e and U_k are equivalent half planes under G, then there is an element in G which fixes ∞ and maps U_e into U_k, hence it must be an elliptic element of order two, let it be $B(z)=-z+s$. Then we have $gB(z)=-\bar{z}+b+s$, a special case of (7). Now if the above two half planes are inequivalent under G, then by the fact that π_1 is a two-to-one map, we conclude that $\pi_0(U_e)$ satisfies (5). Since π_1 is two-to-one, if there is an element of the form (9) then $\pi_1(U_e)$ is completely determined by the subgroup generated by A and g, and $U_e/\{A, g\}$ is conformally equivalent to the punctured half disc and satisfies (6).

LEMMA 3. *Let E be a (non-elementary) extended group and let p be a non-limit point of E. Then the subgroup E_p which consists of all elements fixing p, is either generated by an elliptic element of finite order or generated by an elliptic element of finite order and an anti-analytic element of order two.*

PROOF. Let $E=G\cup GU$ and $G_p=G\cap E_p$. Then it is known that G_p is generated by an elliptic element of finite order (see Kra [9]). Now if E_p contains an anti-analytic element V, then $E_p=G_p\cup G_pV$. Without loss of generality, we can assume that elliptic elements in G_p fix 0 and ∞.

Let $p=\infty$, then we have

(10) $\qquad V^2=e^{i\theta}z,\ e^{i\theta}z\in G_p.$

Then by a simple calculation, we know that $\theta=0$ and V fixes 0 and ∞. Hence V is an element of order two.

Let D be the region of discontinuity of E and let G be the maximal Kleinian group contained in E. Let $\pi_0\colon D\to D/G$ and $\pi\colon D\to D/E$ be the natural projection maps then $\pi_0(\pi)$ is locally one-to-one except at the points which have nontrivial stabilizer subgroup $G_z(E_z)$. If at $zG_z(E_z)$ has order n then $\pi_0(\pi)$ is n-to-1.

Let D be as the above then we say $D/G(D/E)$ is of *finite type* if $D/G(D/E)$

consists of finite numbers of connected Riemann (Klein) surfaces, D contains a finite number of inequivalent elliptic fixed points and each component of $D/G(D/E)$ is obtained by subtracting a finite number of points from a compact Riemann (Klein) surface (with boundary).

For $z\in D$ if the order of $G_z=G\cap E_z$ is n then we call z an *elliptic fixed point of order n.*

Let D_j be a connected component of the region of iscontinuity of E and let E_j be the group consisting of all elements of E which map D_j into itself. Let $G_j=E_j\cap G$.

We shall say that $D_j/G_j(D_j/E_j)$ has *signature* $I=\{X;\ a_1, a_2, \cdots, a_n\}$, if D_j/G_j (D_j/E_j) has Euler characteristic $(-x)$, D_j contains k inequivalent elliptic fixed points of order $a_1, a_2, \cdots, a_k$ respectively and $D_j/G_j(D_j/E_j)$ misses exactly $n-k$ points from a compact Riemann (Klein) surface, where $a_{k+1}, =, \cdots, =a_n=\infty$.

Let D be an invariant union of components of the region of discontinuity of E, then we define

$$\text{Area}(D/G)=2\iint_{D/G}\lambda_D^2(z)\,|dz\wedge d\bar{z}|.$$

Let D/G be a single Riemann surface with signature $\{x;a_1, ;\cdots, a_n\}$, then the following is well known, see Kra [12],

(11) $$\text{Area }(D/G)=2\pi\{x+\sum_{i=1}^{n}(1-1/a_i)\}.$$

We define

$$\text{Area }(D/E)=2\pi\iint_{D/E}\lambda_D^2(z)\,|dz\wedge d\bar{z}|.$$

Let D/G be a single Riemann surface of finite type and assume the natural projection π_1: $D/G\to D/E$ is two-to-one.

Then we have

(12) $$\text{Area }(D/E)=\frac{1}{2}\text{ Area }(D/G).$$

In case (12) suppose D/E has signature

$$\{x;\ a_1, a_2, \cdots, a_s, a_{s+1}, \cdots, a_t\}.$$

Let $p_k\in\overline{D/E}$ correspond to a_k and suppose only $\{P_{s+1}, \cdots, P_t\}$ lie on the boundary of $\overline{D/E}$. Then D/G has signature

(13) $$\{2x;a_{11}, a_{12}, \cdots, a_{s2}, a_{s+1}, \cdots, a_t\},$$

where $a_{ij}=a_i$ for $(i=1, \cdots, s)$ and $(j=1, 2)$.

LEMMA 4. *Let D be an invariant union of the components of the region of discontinuity, let D/E be a single Klein surface with signature*

$\{x; a_1, a_2, \cdots, a_s, a_{s+1}, \cdots, a_t\}$,

and suppose only $a_{s+1}, \cdots, a_t$ *correspond to boundary points of* $\overline{D/E}$. *Then we have the following identity*:

(14) $$Area\ (D/E) = 2\pi\{x + \sum_{i=1}^{s}(1-1/a_i) + \frac{1}{2}\sum_{i=s+1}^{t}(1-1/a_i)\}.$$

PROOF. Compare (11), (12) and (13).

Let D be as in the above lemma. Henceforth we assume D/E is of finite type. Let

$$\pi_1: D/G \to D/E$$

be the natural projection. We know that π_1 is locally one-to-one except at the points which lie over the boundary of D/E, and over the boundary it is two-to-one. Hence, for any point $t \in D/G$ and $\pi_1(t) = p \in D/E$, we can choose local uniformizers T and Z at t and p respectively such that

(15) $$Z \circ \pi_1 \circ T^{-1} = \text{identity}$$

where it is defined. The relation (15) can be extended to $\overline{D/G}$ and $\overline{D/E}$.

Let $f \in A_q^\infty(D, E)$ then we set

$$f(z) = F(Z)\left(\frac{dZ}{dz}\right)^q, \quad z \in D$$

where Z is a local parameter at $\pi(z)$ on D/E. Then F is a q-differential on D/E. By (15), as an application of the Kleinian group case, we know that F can be extended to $\overline{D/E}$. Let G_z be the stabilizer group of z as before, and let a be the order of G_z; then a is a function on D/E. We define $a(p) = \infty$ for $p \in (\overline{D/E} - D/E)$. For any real x we define $[x]$ as the largest integer not greater than x; then, as an application of the Kleinian group case (see Kra [9]), we have the following two identities

(16) $$\text{order}\ _pF \geq -[q(1-1/a(p))], \quad 0 < a(p) < \infty$$

and

(17) $$\text{order}\ _pF \geq 1-q, \quad \text{if } a(p) = \infty.$$

Following Kra [9], we consider the following divisor

(18) $$d(q) = \sum_{p \in \overline{D/E}} \eta_q(p)p$$

where $\eta_q(p) = [q(1-1/a(p))]$ if $a(p) < \infty$ and $\eta_q(p) = 1-q$ if $a(p) = \infty$. Let F be the q-differential defined above, then

$$\text{divisor } F - d(q) \geq 0.$$

Conversely a meromorphic q-differential on $\overline{D/E}$ which satisfies the above lifts to a q-form $f \in A_q(D,E)$.

THEOREM 4. *Let D be an invariant union of components of the region of discontinuity of E, let D/E be of finite type and assume it has k-connected components. Let q be an integer, $q \geq 2$ and $p \geq 1$. Then we have the following identities:*

(19) $$A_q^{\infty}(D,E) = A_q^p(D,E) < \infty$$

(20) $$A_q^{\infty}(D,E) = \prod_{j=1}^{k} A_q(D_j, E_j)$$

(21) $$\dim A_q^{\infty}(D_j, E_j) = (2q-1)x + 2\sum_{i=1}^{s} [(q-q/a_i)] + \sum_{i=s+1}^{t} [(q-q/a_i)]$$

where D_j/E_j has signature

$$\{x; a_1, a_2, \cdots, a_s, a_{s+1}, \cdots, a_t\}$$

and $a_{s+1}, \cdots, a_t$ corresponds to boundary points of $\overline{D/E}$; if $a_i = \infty$ then interpret $[(q-q/a_i)] = q-1$.

PROOF. Since $A_q^p(D,E) \subset A_q^p(D,G)$, and (19) is true for the Kleinian group case, hence it is true for our case also. (20) is clear. We prove (21) following Kra [9]. Let M be a divisor on D_j/E_j such that

$$M = \sum_{i=1}^{s} m_i p_i + \sum_{j=1}^{t} n_j q_j$$

where q_j (boundary of $\overline{D_j/E_j}$) for $j=1,2,\cdots,t$ and $p_i \notin$(boundary of $\overline{D_j/E_j}$) for $i=1,2,\cdots,s$. Set

$$\overline{M} = \sum_{i=1}^{s} 2m_i p_i + \sum_{j=1}^{t} n_j q_j.$$

Call $\overline{M}$ the *double* of M. Let W be a linear differential on $\overline{D_j/E_j}$ and let w be the divisor of W and let (f) be the divisor of a meromorphic function f on $\overline{D_j/E_j}$. Set $a = qw - d(q)$ and let $l_q(a) = \{f : (f) + a \geq 0\}$; then by a simple calculation we have

$$l_q(a) \cong A_q^{\infty}(D_j, E_j).$$

Let $\bar{a}$ be the double of a then by the Riemann-Roch Theorem (see [12]) we have

$$\dim l_q(a) = \deg \bar{a} + \dim(w-a) - x.$$

$\deg \bar{a} = 2qx + 2\sum_{i=1}^{s} [(q-q/a_i)] + \sum_{i=s+1}^{k} [(q-q/a_i)]$, and by an application of the Kleinian group case, we have $\deg(w-a) = 0$, see Kra [9]. Hence we have $\dim(w-a) = 0$

and we have the theorem.

Following Bers [5], we have the following coroll ry.

COROLLARY 5. *D/E is of finite type then*

$$\frac{1}{q}\pi \lim_{q\to\infty} A_q(D,E)=\text{Area}(D/E).$$

COROLLARY 6. *G be the maximal Kleinian group contained in E then we have*

$$\text{Complex dim } A_q(D,G)=\text{Real dim } A_q^{\infty}(D,E).$$

Let E and G be as usual. We observe that E is finitely generated if and only if G is. Also we know D/E is of finite type if and only if D/G is of finite type. Ahlfors' finiteness theorem says that if G is finitely generated then D/G is of finite type. It is clear that Ahlfors' finiteness theorem extends to the extended group case also. For the Ahlfors' finiteness theorem, see Ahlfors [1], Bers [6], Ahlfors [2] and Kra [9].

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