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ON SOME FIXED POINT THEOREMS

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1.1 Introduction.

In [1] Ray has given a theorem on fixed point of mappings in a metric space. Theorem is the following.

THEOREM A. *Let* T_1 *and* T_2 *be maps, each mapping a complete metric space* (E, d) *into itself such that*

(i) $d(T_1x, T_2y) \leq \alpha d(x, y)$; *where* $0<\alpha<1$ *and* x, y *belong to* E $(x \neq y)$ *and*

(ii) *there is a point* x_0 *in* E *such that any two consecutive members of* $\{x_1=T_1x_0, x_2=T^{2}x_1, x_3=T_1x_2, x_4=T_2x_3, \cdots\}$ *are distinct, then* T_1 *and* T_2 *have a unique common fixed point in* E.

We give below the definition of ε-chainable metric space as in reference [2].

DEFINITION. A metric space (E, d) will be said to be *ε-chainable* if for every x, y belonging to E there exists an ε-chain, i.e. a finite set of points $x=x_0, x_1, x_2, \cdots, x_n=y$ (n may depend both on x and y), such that $d(x_{i-1}, x_i)<\varepsilon$, $(i=1, 2, \cdots, n)$.

We prove next the following theorem.

THEOREM 1.1. *Let* E *be a complete* ε*-chainable metric space; and let* T_1 *and* T_2 *be two maps each mapping* E *into itself such that if* $0<d(x, y)<\varepsilon$, *then,*

(i) $d(T_ix, T_iy) \leq \alpha d(x, y)$; $i=1, 2$.

(ii) $d(T_ix, T_iy) \leq \alpha d(x, y)$; $i \neq j$.

where in (i) *and* (ii) x, y *belong to* E $(x \neq y)$ *and* $0<\alpha<1$. *Also* T_1 *and* T_2 *satisfy the condition* (ii) *of theorem* A : *then* T_1 *and* T_2 *have a common fixed point in* E.

PROOF. Since (E, d) is ε-chainable we define for x, y belonging to E

$$d_\varepsilon(x, y) = \inf \sum_{i=1}^{n} d(x_i, x_{i-1}),$$

where the infimum is taken over all ε-chains $x_0, x_1, x_2, \cdots, x_n$ joining $x_0=x$ and $x_n=y$. Then d is a distance function in E satisfying:

(1) $d(x, y)<d_\varepsilon(x, y)$

(2) $d(x, y)=d_\varepsilon(x, y)$ for $d(x, y)<\varepsilon$.

From (2) it follows that a sequence $\{x_n\}$, $x_n\in E$ is a cauchy sequence with respect to d_ε if and only if it is a cauchy sequence with respect to d and is convergent with respect to d_ε if and only if it is convergent with respect to d. Since (E, d) is complete therefore (E, d_ε) is a complete metric space. Moreover the following is true.

Given x, y in E and any ε-chain $x_0, x_1, x_2, \cdots, x_n$ with $x_0=x$ and $x_n=y$ such that $d(x_i, x_{i-1})<\varepsilon$ $(i=1, 2, \cdots, n)$ we have (if n is even, say $n=2$)

$$d(T_1x_0, T_1x_1)\leq\alpha d(x_0, x_1)<\varepsilon$$

$$d(T_2x_2, T_1x_1)\leq\alpha d(x_2, x_1)<\varepsilon$$

so that T_1x_0, T_1x_1, T_2x_2 form an ε-chain for T_1x_0 and T_2x_2. Similarly if n is odd(say $n=3$) we can show that T_1x_0, T_2x_1, T_1x_2, T_2x_3 form an ε-chain for T_1x_0, T_2x_3. Condition (i) is also necessary which can be seen by taking $n=4$.
Combining all the cases above it can be shown that

$$d_\varepsilon(T_1x, T_2y)\leq\alpha\sum_{i=1}^{n}d(x_{i-1}, x_i)$$

$x_0, x_1, x_2, \cdots, x_n=y$ being an arbitrary ε-chain,
therefore we have

$$d_\varepsilon(T_1x, T_2y)\leq\alpha d_\varepsilon(x, y),$$

and since T_1 and T_2 also satisfy the condition (ii) of theorem A therefore by the conclusion of theorem A we have the required result.

THEOREM 1.2. *Let E be a complete ε-chainable metric space; and let T_1 and T_2 be two maps each mapping E into itself. If there exists two integers p_1 and p_2 such that if* $0<d(x, y)<\varepsilon$, *then*

(i) $d(T_i^{p_i}x, T_i^{p_i}y)\leq\alpha d(x, y)$; $i=1, 2$;

(ii) $d(T_i^{p_i}x, T_j^{p_j}y)\leq\alpha d(x, y)$: $i\neq j$;

where in (i) *and* (ii) $x, y\in E$ $(x\neq y)$ *and* $0<\alpha<1$. *Also let $T_1^{p_1}$ and $T_2^{p_2}$ satisfy the condition* (ii) *of theorem A, then T_1 and T_2 have a common fixed point.*

PROOF. Set $S_1=T_1^{p_1}$ and $S_2=T_2^{p_2}$. Then by theorem 1.1 there exists a unique fixed point x such that $S_1(x)=S_2(x)=x$., i.e., $T_1^{p_1}(x)=x=T_2^{p_2}(x)$. From which it follows that $T_1^{p_1+1}(x)=T_1(x)$, which implies that $T_1^{p_1}(T_1(x))=T_1(x)$, since $T_1^{p_1}$

has a unique fixed point, therefore $T_1(x)=x$. Similarly it follows that $T_2(x)=x$.

THEOREM 1.3. *Let E be a complete ε-chainable metric space. Let T_1 and T_2 be two mappings of E into itself and suppose there exists a mapping K of E into itself such that K has a right inverse K^{-1}(i.e., a function K such that $KK^{-1}=I$, where I is the identity mapping of E) and if $0<d(x, y)<\varepsilon$, then*

(i) $d(K^{-1}T_iKx,\ K^{-1}T_iKy)\leq\alpha d(x, y)$; $i=1,2$;

(ii) $d(K^{-1}T_iKx,\ K^{-1}T_jKy)\leq\alpha d(x, y)$; $i\neq j$;

where in (i) *and* (ii) *$x, y\in E$ $(x\neq y)$ and $0<\alpha<1$. Also suppose $K^{-1}T_1K$ and $K^{-1}T_2K$ satisfy the condition* (ii) *of theorem A. Then T_1 and T_2 possess a common fixed point which is unique.*

PROOF. Set $K^{-1}T_1K=S_1$ and $K^{-1}T_2K=S_2$, then S_1 and S_2 have a common fixed point which is unique, by theorem 1.1. i.e., $K^{-1}T_1K(x)=x=K^{-1}T_2K(x)$ from which we get, $KK^{-1}T_1K(x)=K(x)$, therefore $T_1(K(x))=K(x)$. Similarly $T_2(K(x))=K(x)$, in other words T_1 and T_2 have a common fixed point $K(x)$.

Next we give a theorem on sequence of mappings and their fixed points.

THEOREM 1.4. *Let (E, d) be a complete metric space and let T_k^1 and T_k^2 be two sequences of mappings each mapping E into itself such that*

$$d(T_k^1x,\ T_k^2y)\leq\beta_k\{d(T_k^1x,\ x)+d(T_k^2y,\ y)\}$$

where all β_k's are $<\frac{1}{2}$ and positive, and $x, y\in E$ $(x\neq y)$ $(k=1,2,\cdots)$. Let T^1 and T^2 be mappings such that $\lim_{k\to\infty} d(T_k^1x, T^1x)=0$ and $\lim_{k\to\infty} d(T_k^2x,\ T^2x)=0$, for all x in E. Also $\beta_k\to\beta$ $(0<\beta<\frac{1}{2})$ as $k\to\infty$. Then T^1 and T^2 have a common fixed point (say u). If u_k for a fixed k is the simultaneous fixed point of T_k^1 and T_k^2 (which exists because of a theorem in [4]) *then $\lim_{k\to\infty} u_k=u$.*

PROOF. To prove the first part of the theorem we have only to show that T^1 and T^2 satisfy the inequality (*) as given below.

Now,

$$d(T^1x,\ T^2y)\leq d(T^1x, T_k^1x)+d(T_k^1x, T_k^2y)+d(T_k^2y, T^2y)$$

$$\leq d(T^1x,\ T_k^1x)+d(T_k^2y,\ T^2y)+\beta_k\{d(T_k^1x,\ x)+d(T_k^2y,\ y)\}$$

$$\leq d(T^1x,\ T_k^1x)+d(T_k^2y,\ T^2y)+\beta_k\{d(T_k^1x,\ T^1x)$$

$$+d(T^1x,\ x)+d(T_k^2y,\ T^2y)+d(T^2y,\ y)\}$$

therefore as $k\to\infty$, since

$$d(T_k^1x,\ T^1x)\to 0 \text{ and also } d(T_k^2x,\ T^2x)\to 0$$

we have

$$d(T^1x,\ T^2y)\leq\beta\{d(T^1x,\ x)+d(T^2y,\ y)\} \qquad (*)$$

Then by the theorem given in [3] T^1 and T^2 both have a simultaneous fixed point in E (which we denote by u say). Now to prove that $\lim_{k\to\infty} u_k=u$ we proceed as follows: Since u is in E, fixing $n=n_0$ we form the following sequence

$$x_1=T_{n_0}^1(u),\ x_2=T_{n_0}^2(x_1),\ x_3=T_{n_0}^1(x_2),\ x_4=T_{n_0}^2(x_3),\ \cdots$$

then after little calculation as done in [4], it can be shown that

$$d(x_k,\ x_{k+1})\leq\left(\frac{\beta_{n_0}}{1-\beta_{n_0}}\right)^k d(u,\ T_{n_0}^1(u))$$

from this it follows that

$$d(x_k,\ x_{k+p})\leq\frac{r^k}{1-r}d(u,\ T_{n_0}^1(u))$$

where $r=\dfrac{\beta_{n_0}}{1-\beta_{n_0}}$,

therefore there exists u_{n_0} such that $\lim_{k\to\infty} x_k=u_{n_0}$. Now to show that

$$T_{n_0}^1(u_{n_0})=u_{n_0}=T_{n_0}^2(u_{n_0}).$$

We need the following inequality

$$d(u_{n_0},\ T_{n_0}^1(u_{n_0}))\leq d(u_{n_0},\ x_k)+d(x_k,\ T_{n_0}^1(u_{n_0}))$$

$$=d(u_{n_0},\ x_k)+d(T_{n_0}^2(x_{k-1}),\ T_{n_0}^1(u_{n_0})),$$

where we choose k to be even positive integer.

Therefore

$$d(u_{n_0},\ T_{n_0}^1(u_{n_0}))\leq d(u_{n_0},\ x_k)+\beta_{n_0}\left\{d(x_{k-1},\ T_{n_0}^2(x_{k-1}))+d(u_{n_0},\ T_{n_0}^1(u_{n_0}))\right\}$$

$$\text{i.e., } (1-\beta_{n_0})d(u_{n_0},\ T_{n_0}^1(u))\leq d(u_{n_0},\ x_k)+\beta_{n_0}d(x_{k-1},\ x_k)$$

and letting $k\to\infty$, we can prove that $T_{n_0}^1(u_{n_0})=u_{n_0}$. That having proved we start with the following inequality,

$$d(u_{n_0}, T^2_{n_0}(u_{n_0})) \leq d(u_{n_0}, x_1) + d(x_1, T^2_{n_0}(u_{n_0}))$$
$$= d(u_{n_0}, T^1_{n_0}(u)) + d(T^1_{n_0}(u), T^2_{n_0}(u_{n_0}))$$
$$\leq d(u_{n_0}, T^1_{n_0}(u)) + \beta_{n_0}\{d(u, T^1_{n_0}(u)) + d(u_{n_0}, T^2_{n_0}(u_{n_0}))\}$$

i.e.,

$$(1-\beta_{n_0})d(u_{n_0}, T^2_{n_0}(u_{n_0})) \leq d(u_{n_0}, T^1(u)) + d(T^1(u), T^1_{n_0}(u)) + \beta_{n_0} d(u, T^1_{n_0}(u))$$

i.e.,

$$(1-\beta_{n_0})d(u_{n_0}, T_{n_0}(u_{n_0})) \leq d(u_{n_0}, u) + d(T^1(u), T^1_{n_0}(u)) + \beta_{n_0} d(T(u), T^1_{n_0}(u))$$

therefore as $n_0 \to \infty$, $d(u_{n_0}, u) \to 0$; which completes the proof.

2.1 In this section we give certain fixed point theorems on a generalized complete metric space. We give below the definition and the characterization of a generalized complete metric space as given in [5]. Theorem B and C mentioned below have also been given in [5].

DEFINITION. The pair (E, d) is called a *generalized complete metric space* if E is a non-void set and d is a function from $E \times E$ to extended real numbers satisfying the following conditions:

(D0) $d(x, y) \geq 0$

(D1) $d(x, y) = 0$ iff $x = y$

(D2) $d(x, y) = d(y, x)$

(D3) $d(x, y) \leq d(x, z) + d(z, y)$

(D4) every d-cauchy sequence in E is d-convergent, i.e.,

if $\{x_n\}$ is a sequence in E such that $\lim_{m, n \to \infty} d(x_n, x_m) = 0$, then there is an $x \in E$ with $\lim_{n \to \infty} d(x_n, x) = 0$. For convenience we will say that a pair (E, d) is a generalized metric space if all but (D4) of the above conditions are satisfied. Let $\{(E_\alpha, d_\alpha) | \alpha \in O\}$ be a family of disjoint metric spaces. Then there is a natural way of getting a generalized metric space (E, d) from $\{(E_\alpha, d_\alpha) | \alpha \in O\}$ as follows.

For any $x, y \in E$ define

$$d(x, y) = d_\alpha(x, y) \text{ if } x, y \in E_\alpha \text{ for some } \alpha \in O$$
$$= +\infty \quad \text{if } x \in E_\alpha \text{ and } y \in E_\beta \text{ for some } \alpha, \beta \in O \text{ with } \alpha \neq \beta.$$

Clearly (E, d) is a generalized metric space. moreover if (E_α, d_α) is complete then (E, d) is a generalized complete metric space. The main purpose of the above procedure is to show that the above method is the only way to obtain generalized complete metric spaces.

Let (E, d) be a Generalized Complete Metric Space. Define $\sim$ on E as follows, $x \sim y$ iff $d(x, y) < \infty$. Then $\sim$ is an equivalence relation on E. Therefore E is decomposed (uniquely) into disjoint equivalence classes E_α, $\alpha \in O$. From henceforth we will reserve the term 'Canonical decomposition' for the type of decomposition as shown above.

THEOREM B. *Let (E, d) be a generalized metric space. $E = \bigcup \{E_\alpha | \alpha \in O\}$ the canonical decomposition and $d_\alpha = d | E_\alpha \times E_\alpha$ for each $\alpha \in O$. Then*

(a) *for $\alpha \in O$, (E_α, d_α) is a metric space.*

(b) *for any $\alpha, \beta \in O$ with $\alpha \neq \beta$, $d(x, y) = +\infty$ for any $x \in E_\alpha$ and $y \in E_\beta$.*

(c) *(E, d) is a generalized complete metric space iff for each $\alpha \in O$, (E_α, d_α) is a complete metric space.*

THEOREM C. *Let (E, d) be a generalized metric space. $E = \bigcup \{E_\alpha | \alpha \in O\}$ the canonical decomposition and let $T : E \to E$ be a mapping such that $d(T(x), T(y)) < \infty \cdots\cdots (*)$ whenever $x, y \in E$ and $d(x, y) < \infty$. Then T has a fixed point iff $T_\alpha = T | E_\alpha : E_\alpha \to E_\alpha$ has a fixed point for some $\alpha \in O$.*

We note that (*) is necessary for T to be a mapping from $E_\alpha \to E_\alpha$.
We prove next the following theorems.

THEOREM 2.1. *Let (E, d) be a generalized complete metric space ; $E = \bigcup \{E_\alpha | \alpha \in O\}$ be the canonical decomposition. Let $T_1 : E \to E$ and $T_2 : E \to E$ be two mappings such that*

(a) *$d(T_1 x, T_2 y) \leq \beta \{d(x, T_1 x) + d(y, T_2 y)\}$ for all $x, y \in E$ and $0 < \beta < \frac{1}{2}$,*

(b) *$d(T_i x, T_i y) \leq d(x, y)$ for all $x, y \in E$ $(x \neq y)$, $i = 1, 2$*

$d(T_i x, T_j y) \leq d(x, y)$ for all $x, y \in E (x \neq y)$, $i \neq j$;

if there exists an $x_0 \in E$ such that $d(x_0, T_i(x_0)) < \infty$ for $i = 1$ or 2 then for some $\alpha \in O$, the restrictions

$$T_{1\alpha} : T_1 | E_\alpha : E_\alpha \to E_\alpha \text{ and } T_{2\alpha} : T_2 | E_\alpha : E_\alpha \to E_\alpha$$

satisfy the condition (a) *above.*

PROOF. Let $d(x_0, T_1(x_0))<\infty$, then both x_0 and $T_1(x_0)\in E_{\alpha_0}$ for some $\alpha_0\in O$. Because of (b) we have if $x_1\in E_{\alpha_0}$

$$d(T_1x_1, T_1x_0)\leq d(x_1, x_0)<\infty$$

also

$$d(T_2x_1, T_1x_0)\leq d(x_1, x_0)<\infty$$

and therefore $T_1(E_{\alpha_0})\subset E_{\alpha_0}$ and $T_2(E_{\alpha_0})\subset E_{\alpha_0}$. From which it follows that $T_{1\alpha_0}$ and $T_{2\alpha_0}$ satisfy the condition (b) in E_{α_0}.

THEOREM 2.2. *Assuming the same type of hypothesis as in theorem* 2.1 *above let $x\in E$ and consider the sequence of successive approximation*

$$x_1=x,\ T_1(x_1)=x_2,\ x_3=T_2(x_2),\ x_4=T_1(x_3),\ x_5=T_2(x_4),\cdots.$$

Then the following alternative holds: either

(A) *for every $m=0, 1, 2, \cdots$, one has $d(x_m, x_{m+1})=+\infty$*

or (B) *the sequence $\{x_m\}$ is d-convergent to a simultaneous fixed point of T_1 and T_2.*

PROOF. If (A) does not hold then for some m

$$d(x_m, x_{m+1})<\infty$$

letting $x_m=x_0$, the theorem 2.1 shows that $T_1(E_\alpha)\subset E_\alpha$ and $T_2(E_\alpha)\subset E_\alpha$ where E_α is the complete metric space containing x_0. Therefore by the theorem mentioned in section 1 of the present note also given in [4], the sequence $x_m, x_{m+1}, \cdots$ is d-convergent to a simultaneous fixed point of $T_{1\alpha}$ and $T_{2\alpha}$. This implies that (B) holds.

COROLLARY. *Assuming the same hypothesis for the case of mappings $T_1=T_2=T$(say), let $x\in E$ and consider the sequence of successive approximation with initial value x,*

$$x,\ Tx,\ T^2x,\ T^3x,\cdots,\ T^nx,\cdots$$

then following alternative holds, either

(A) *for every $m=0, 1, \cdots$, one has $d(T^mx, T^{m+1}x)=+\infty$*

or (B) *the sequence, $x, Tx, T^2x, \cdots, T^mx, \cdots$ is d-convergent to a fixed point of T.*

PROOF. Proof is in the same line as in theorem 2.2.

Suppose T be a mapping from a metric space E into itself. Also suppose T satisfies the conditions (A0) and (A1) below.

(A0) $$d(Tx, Ty) \leq \alpha d(x, Tx) + \beta d(y, Ty) + \gamma d(x, y)$$
where $0 < \alpha + \beta + \gamma < 1$.

(A1) $$d(Tx, Ty) \leq d(x, y), \text{ for all } x, y \in E (x \neq y)$$

then from a theorem of Reich [6] it follows that such a T has a unique fixed point in E.

THEOREM 2.3. *Let E be a generalized complete metric space. Suppose T be a mapping from E into itself such that it satisfies conditions* (A0) *and* (A1) *above. Let $x \in E$ and consider the sequence of successive approximation*:

$$x_1 = x, \ Tx_1 = x_2 ; T^2 x_1 = x_3 ; T^3 x_1 = x_4, \cdots$$

then the following alternative holds, either

(a) *for every $m = 0, 1, 2, \cdots$, one has $d(x_m, x_{m+1}) = \infty$*

or

(b) *the sequence $\{v_m\}$ is d-convergent to a fixed point of T.*

PROOF. Proof is similar to the proof of theorem 2.2 above; only in this case we apply finally the theorem of Reich [6].

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REFERENCES

[1] Ray, B. *On a paper of Kannan*, Bull. Cal. Math. Soc. 63, pp7—10(1971)

[2] Edelstein, M, *An extension of Banach's contraction principle*, Proc. Amer. Math. Soc. 12, No. 1, 7—10, 1961

[3] Edelstein, M, *On fixed and periodic points under contractive mapping*, Journal London Math, Soc. 37, 74—79, 1962.

[4] Kannan, R, *On a fixed point theorem in a metric space*, Bull. Cal. Math, Soc. No. 1&2. pp. 71—76, 1968.

[5] Jung, Calvin F. K., *On Generalized Complete Metric Spaces*, Bull. Am. Math. Soc. p. 113, Vol 75, 1969.

[6] Reich, S, *Kannan's fixed point theorem*, Boll. Unione. Math. Italy. 4(1971). 1—11.