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A DIFFERENTIABLE MANIFOLD WITH f-STRUCTURE OF RANK r

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Introduction.

Let us consider an n-dimensional real differentiable manifold V_n of differentiability class C^{r+1}. Let there exist in V_n a vector valued linear function f satisfying

(1.1) a $\qquad \overline{\overline{X}}+\overline{X}=0,$ for an arbitrary vector field X,

where

(1.1) b $$\overline{X} \overset{\text{def}}{=} f(X),$$

and

(1.2) $$\text{rank}(f)=r,$$

is constant everywhere. Then f is called an *f-structure of rank r* and V_n is called an *n-dimensional manifold with a f-structure of rank r.*

AGREEMENT (1.1). *In the above and in what follows, the equations containing X,Y,Z,U hold for arbitrary vector field X,Y,Z,U etc.*

The eigen values of f are given by

(1.3) $$\lambda^{n-2m}(\lambda+i)^m(\lambda-i)^m=0.$$

Three cases arise

Case 1. $\text{rank}(f)=n$.

In this case (1.1)a reduces to

(1.4) $$\overline{\overline{X}}+X=0.$$

n is even $=2m$ and the eigen values are given by $(\lambda+i)^{\frac{1}{2}n}(\lambda-i)^{\frac{1}{2}n}=0$. V_n is said to be an almost complex manifold and f is called an almost complex structure.

Case 2. $\text{rank}(f)=n-1$.

In this case, one of the eigen values is 0, the corresponding eigen vector being T, such that $\overline{T}=0$. Consequently from (1.1)a, we have

(1.5) $$\overline{\overline{X}}+X=A(X)T,$$

where A is a 1-form. n is odd, say $n=2m+1$ and the eigen values are given by $\lambda(\lambda+i)^{\frac{n-1}{2}}(\lambda-i)^{\frac{n-1}{2}}=0$. V_n is called an almost contact manifold and the structure (f,T,A) is called an almost contact structure.

Case 3. $\text{rank}(F)=r,\ 1\leq r\leq n-1$.

In this case there are $n-r$ repeated eigen values 0, corresponding to which there is a pencil of eigen vectors. Let $\underset{x}{T}$, $1\leq x\leq n-r$ be a linearly independent set of eigen vectors corresponding to the eigen values 0. Then

(1.6) $$\overline{\underset{x}{T}}=0.$$

Hence from (1.1)a, we have

(1.7) $$\overline{\overline{X}}+X=\underset{x}{A}(X)\underset{x}{T},$$

where $\underset{x}{A}$ are arbitrary 1-forms. r is even$=2m$ say. The eigen values are given by

$$\lambda^{n-2m}(\lambda+i)^{m}(\lambda-i)^{m}=0.$$

Barring X in (1.7) and comparing the resulting equation with (1.1)a, we get

(1.8) $$\underset{x}{A}(\overline{X})=0.$$

Barring (1.6) and using (1.7), we get

(1.9) $$\underset{y}{A}(\underset{x}{T})=\delta_{yx}$$

Let us put

(1.10) a) $l(X)=-\overline{\overline{X}}$, b) $m(X)=\overline{\overline{X}}+X$.

Then it can be easily proved that

(1.11) a) $l^2(X)\overset{\text{def}}{=}l(l(X))=l(X)$, b) $m^2(X)=m(X)$,

c) $l(m(X))=m(l(X))=0$, d) $l(\overline{X})=\overline{l(X)}=\overline{X}$,

e) $m(\overline{X})=\overline{m(X)}=0$, f) $l(\overline{\overline{X}})=\overline{\overline{l(X)}}=-l(X)$,

g) $X=l(X)+m(X)$, h) $\text{rank}(l)=r$,

i) $\text{rank}(m)=n-r$.

Thus the operators l and m applied to the tangent space at each point of the manifold are complementary projection operators. There exist two complementary distributions Π_r and Π_{n-r} corresponding to l and m respectively, such that Π_r and Π_{n-r} are r and $(n-r)$ dimensional.

2. Eigen vectors.

THEOREM (2.1). *The eigen values of l are given by*

(2.1) $$\lambda^{n-r}(\lambda-1)^{r}=0.$$

Let X be an arbitrary vector. Then $\overline{X}$ is in the pencil of eigen vectors corresponding to the eigen value 1 *and $\overline{\overline{X}}+X$ is in the pencil of eigen vectors corresponding to the eigen value* 0.

PROOF. Let λ be an eigen value of l and p be the corresponding eigen vector. Then

$$l(P)=\lambda P,\quad l^2(P)=\lambda^2 P$$

Plugging in these in (1.11)a, we get (2.1). Remaining part of the statement follows from (1.11)d and (1.11)c and (1.10)b.

THEOREM (2.2). *The eigen values of m are given by*

$$\lambda^r(\lambda-1)^{n-r}=0. \tag{2.2}$$

Let X be an arbitrary vector. Then $\overline{X}$ *is in the pencil of eigen vectors corresponding to the eigen value* 0 *and* $\overline{\overline{X}}+X$ *is in the pencil of eigen vectors corresponding to the eigen value* 1.

The proof follows the pattern of the proof of Theorem (2.1).

THEOREM (2.3). *Let* $\begin{cases}P\\Q\end{cases}$ *be an eigen vector of l corresponding to the eigen value* $\begin{cases}0\\1.\end{cases}$ *Then* $\begin{cases}\overline{P}=0\\ \overline{\overline{Q}}+Q=0.\end{cases}$ *Consequently* $\begin{cases}P\\Q\end{cases}$ *is an eigen vector of f corresponding to the eigen value* $\begin{cases}0\\ \pm i.\end{cases}$

PROOF. Since P is an eigen vector of l corresponding to the eigen value 0, $l(P)=0\Leftrightarrow-\overline{\overline{P}}=0\Leftrightarrow\overline{P}=0$.

Since Q is an eigen vector of l corresponding to the eigen value 1, $l(Q)=Q\Leftrightarrow\overline{\overline{Q}}+Q=0$. Remaining part of the proof is obvious.

THEOREM (2.4). *Let* $\begin{cases}p\\q\end{cases}$ *be an eigen vector of m corresponding to the eigen value* $\begin{cases}1\\0.\end{cases}$ *Then* $\begin{cases}\overline{p}=0\\ \overline{\overline{q}}+q=0.\end{cases}$ *Consequently* $\begin{cases}p\\q\end{cases}$ *is an eigen vector of f corresponding to the eigen value* $\begin{cases}0\\ \pm i.\end{cases}$

The proof follows the pattern of the proof of Theorem (2.4).

COROLLARY (2.1). *Let P be an eigen vector of l. Then* $\overline{p}=\overline{\overline{p}}=\cdots=0$.

PROOF. We have

$$l(P)=\lambda P,$$

whence

$$\overline{l(P)}=\lambda\overline{P}.$$

In consequence of (1.11)d, this equation takes the form

$$l(\overline{P})=\lambda\overline{P}.$$

Hence, we have the statement.

3. f-structure.

THEOREM (3.1). *f-structure is not unique. Let μ be a non-singular vector valued linear function in V_n. Then f' defined by*

(3.1) $$\mu(f'(X)) \overset{\text{def}}{=} \overline{\mu(X)},$$

is also f-structure.

PROOF. In consequence of (3.1) and (1.1), we have

(3.2)a $$\overline{\overline{\mu(f'(X))}} = -\overline{\mu(X)} = -\mu(f'(X)).$$

Also from (3.1), we get

(3.2)b $$\overline{\overline{\mu(f'(X))}} = \mu(f'^2(X)) = \mu(f'^3(X)).$$

From (3.2)a, b, we have

$$\mu(f'^3(X) + f'(X)) = 0.$$

Since μ is non-singular,

$$f'^3(X) + f'(X) = 0,$$

which proves the statement.

THEOREM (3.2). *We have*

(3.3) $$\mu(l'(X)) = l(\mu(X)),$$

(3.4) $$\mu(m'(X)) = m(\mu(X)).$$

PROOF. In consequence of (3.1) and (1.10)a, we have

$$\mu(l'(X)) = -\mu(f'^2(X)) = -\overline{\mu(f'(X))} = -\overline{\overline{\mu(X)}} = l(\mu(X)).$$

Hence, we have (3.3). Similarly

$$\mu(m'(X)) = \mu(f'^2(X)) + \mu(X) = \overline{\overline{\mu(X)}} + \mu(X) = m(\mu(X)).$$

In the manifold with f-structure V_n, we can always introduce a metric tensor g. Let g satisfy

(3.5)a $$g(\overline{X}, \overline{Y}) = -g(\overline{\overline{X}}, Y) = -g(X, \overline{\overline{Y}}).$$

We are justified in assuming g as above, because of the following two considerations.

(i) g is symmetric

(ii) Repeated operation of barring X or Y in 3.5)a yields the same set equations and there is no contradiction.

Let us put

(3.6) $$'m(X, Y) = g(m(X), Y) = g(X, m(Y)).$$

Then (1. 17) assumes the form

$$(3.5)\text{b} \qquad g(\bar{X}, \bar{Y}) = g(X, Y) - {}'m(X, Y).$$

The equations (3. 5)a, b are also equivalent to

$$(3.5)\text{c} \qquad g(\bar{\bar{X}}, \bar{\bar{Y}}) = g(\bar{X}, \bar{Y}).$$

We also have

$$(3.5)\text{d} \qquad g(\bar{X}, \bar{\bar{Y}}) + g(\bar{\bar{X}}, \bar{Y}) = 0,$$

which in consequence of (1. 10)b and (3. 6) assumes the form

$$(3.5)\text{e} \qquad g(X, \bar{Y}) + g(\bar{X}, Y) = 0.$$

THEOREM (3. 3). *Let us put*

$$(3.7) \qquad g'(X, Y) = g(\mu(X), \mu(Y)).$$

Then g' also satisfies an equation of the type (3. 5), *that is*

$$(3.8)\text{a} \qquad g'(f'(X), f'(Y)) = g'(X, Y) - {}'m'(X, Y),$$

where

$$(3.8)\text{b} \qquad {}'m'(X, Y) = g'(m'(X), Y) = g'(X, m'(Y)).$$

PROOF. In consequence of (3. 7), (3. 1), (3. 5)b, (3. 6) and (3. 4)

$$\begin{aligned} g'(f'(X), f'(Y)) &= g(\mu(f'(X)), \mu(f'(Y))) \\ &= g(\overline{\mu(X)}, \overline{\mu(Y)}) \\ &= g(\mu(X), \mu(Y)) - {}'m(\mu(X), \mu(Y)) \\ &= g'(X, Y) - g(m(\mu(X)), \mu(Y)) \\ &= g'(X, Y) - g(\mu(m'(X)), \mu(Y)) \\ &= g'(X, Y) - g'(m'(X), Y) \\ &= g'(X, Y) - {}'m'(X, Y). \end{aligned}$$

4. Integrability conditions.

We will prove the following theorem:

THEOREM (4. 1). *The necessary and sufficient condition that V_n be an n-dimensional differentiable manifold with a f-structure of rank 2m is that it contains a distribution Π_m of complex dimension m a distribution $\bar{\Pi}_m$ complex conjugate to Π_m and a distribution Π_{n-2m} of real dimension $n-2m$, such that Π_m, $\bar{\Pi}_m$, Π_{n-2m} have no direction in common and span together a linear manifold of dimension n.*

PROOF. It can be proved by usual method that in an n-dimensional differentiable manifold with a f-structure of rank $2m$, there is a distribution Π_m of complex dimension m, a distribution $\bar{\Pi}_m$ complex conjugate to Π_m and a distribution Π_{n-2m}

of real dimension $n-2m$, such that Π_m, $\tilde{\Pi}_m$ and Π_{n-2m} have no direction in common and span together a linear manifold of dimension n, projections on Π_m, $\tilde{\Pi}_m$ and Π_{n-2m} being given by L, M, m respectively, such that

$$2L(X)=-\overline{\overline{X}}-i\overline{X}, \tag{4.1}$$

$$2M(X)=-\overline{\overline{X}}+i\overline{X}, \tag{4.2}$$

$$m(X)=\overline{\overline{X}}+X. \tag{4.3}$$

That the condition is sufficient can also be proved similarly

Let $\underset{x}{P}, \underset{x}{Q}, \underset{a}{T}$, $1\leq x\leq m$, $x, y\in N$, $1\leq a\leq n-2m$ be the linearly independent eigen vectors corresponding to the eigen values $i, -i, 0$ of f. Then

(4.4) a) $L(\underset{x}{P})=\underset{x}{P}$, b) $L(\underset{x}{Q})=0$, c) $L(\underset{a}{T})=0$,

(4.5) a) $M(\underset{x}{P})=0$, b) $M(\underset{x}{Q})=\underset{x}{Q}$, c) $M(\underset{a}{T})=0$,

(4.6) a) $m(\underset{x}{P})=0$, b) $m(\underset{x}{Q})=0$, c) $m(\underset{a}{T})=\underset{a}{T}$

The equations (4.5), (4.6) and (4.6) can be written in the tabular form as follows.

		$(\underset{x}{P})$	$(\underset{x}{Q})$	$(\underset{a}{T})$
(4.4)	L	$\underset{x}{P}$	0	0
(4.5)	M	0	$\underset{x}{Q}$	0
(4.6)	m	0	0	$\underset{a}{T}$

Now, $\{\underset{x}{P}, \underset{x}{Q}, \underset{a}{T}\}$ is a L. I. set. The inverse set $\overset{x}{p}, \overset{x}{q}, \overset{a}{A}$ is then given by

		$(\underset{y}{P})$	$(\underset{y}{Q})$	$(\underset{b}{T})$
(4.7)	$\overset{x}{p}$	δ^x_y	0	0
(4.8)	$\overset{x}{q}$	0	δ^x_y	0
(4.9)	$\overset{a}{A}$	0	0	δ^a_b

$$\overset{x}{p}(X)\underset{x}{P}+\overset{x}{q}(X)\underset{x}{Q}+\overset{a}{A}(X)\underset{a}{T}=X. \tag{4.10}$$

THEOREM (4.2). *We have*

$$L(X)=\overset{x}{p}(X)\underset{x}{P}, \tag{4.11}$$

(4.12) $$M(X)=\overset{x}{q}(X)\underset{x}{Q},$$

(4.13) $$m(X)=\overset{a}{A}(X)\underset{a}{T}.$$

PROOF. From (4.1) and (4.2), we have

(4.14) $$\overline{X}=i\{L(X)-M(X)\},$$

(4.15) $$\overline{\overline{X}}=-L(X)-M(X).$$

Also from (4.10) we have

(4.16) $$\overline{X}=i\{\overset{x}{p}(X)\underset{x}{P}-\overset{x}{q}(X)\underset{x}{Q}\},$$

(4.17) $$\overline{\overline{X}}=-\overset{x}{p}(X)\underset{x}{P}-\overset{x}{q}(X)\underset{x}{Q}.$$

From the equations (4.14)–(4.17) we have (4.11) and (4.12). In consequence of (4.1), (4.2) and (4.3) we have

(4.18) $$X=L(X)+M(X)+m(X).$$

From (4.10), (4.18), (4.11) and (4.12), we have (4.13).

THEOREM (4.3). *We have*

(4.19)a $$L(M(X))=L(m(X))=M(L(X))=M(m(X))=m(L(X))=m(M(X))=0,$$

(4.19)b $$L^2(X)=L(X),\ M^2(X)=M(X),\ m^2(X)=m(X).$$

PROOF. The proof is obvious.

COROLLARY (4.1). *We have*

(4.19)c $$L(\mu(X))=\mu(L'(X))$$

(4.19)d $$M(\mu(X))=\mu(M'(X))$$

(4.19)e $$m(\mu(X))=\mu(m'(X)).$$

PROOF. We have

$$\begin{aligned}2\mu(L'(X))&=-\mu(f'^2(X))-i\mu(f'(X))=-\overline{\mu(f'(X))}-i\mu(f'(X))\\&=-\overline{\overline{\mu(X)}}-i\overline{\mu(X)}=2L(\mu(X)).\end{aligned}$$

We can similarly prove (4.19)d.

Ishihara and Yano (1964) obtained the integrability conditions for the distributions Π_r, Π_{n-r}. We will obtain the integrability conditions for the distributions Π_m, $\tilde{\Pi}_m$ and Π_{n-2m}. For this we prove the following lemmas.

LEMMA (4.1)a. *We have*

(4.20)a $$2(dL)(m(X),m(Y))=\overline{\overline{N(\bar{X},\bar{Y})}}-\overline{\overline{N(X,Y)}}-\overline{N(\bar{X},Y)}-\overline{N(X,\bar{Y})}$$
$$+i\{\overline{N(\bar{X},\bar{Y})}-\overline{N(X,Y)}+\overline{\overline{N(\bar{X},Y)}}+\overline{\overline{N(X,\bar{Y})}}\},$$

(4.20)b $$2(dM)(m(X),m(Y))=\overline{\overline{N(\bar{\bar{X}},\bar{\bar{Y}})}}-\overline{\overline{N(X,Y)}}-\overline{N(\bar{\bar{X}},Y)}-\overline{N(X,\bar{\bar{Y}})}$$
$$-i\{\overline{N(\bar{\bar{X}},\bar{\bar{Y}})}-\overline{N(X,Y)}+\overline{\overline{N(\bar{\bar{X}},Y)}}+\overline{\overline{N(X,\bar{\bar{Y}})}}\},$$

where N is Nijenhuis tensor given by

(4.21) $$N(X,Y)=[\bar{X},\bar{Y}]+\overline{\overline{[X,Y]}}-\overline{[\bar{X},Y]}-\overline{[X,\bar{Y}]}.$$

PROOF. In consequence of (4.19), we have

$$(dL)(m(X),m(Y))=m(X)(L(m(Y)))-m(Y)(L(m(X)))-L([m(X),m(Y)])$$
(4.22) $$=-L([m(X),m(Y)]).$$

But, by virtue of (4.1), (4.3) and (4.4) we have

(4.23) $$2L([m(X),m(Y)])=2L([\bar{\bar{X}},\bar{\bar{Y}}])+2L([\bar{\bar{X}},Y])+2L(X,\bar{\bar{Y}}])+2L([X,Y])$$
$$=-\{\overline{\overline{[\bar{\bar{X}},\bar{\bar{Y}}]}}+\overline{\overline{[\bar{\bar{X}},Y]}}+\overline{\overline{[X,\bar{\bar{Y}}]}}+\overline{\overline{[X,Y]}}\}$$
$$-i\{\overline{[\bar{\bar{X}},\bar{\bar{Y}}]}+\overline{[\bar{\bar{X}},Y]}+\overline{[X,\bar{\bar{Y}}]}+\overline{[X,Y]}\}$$
$$=-\{\overline{\overline{N(\bar{X},\bar{Y})}}-\overline{\overline{N(X,Y)}}-\overline{N(\bar{X},Y)}-\overline{N(X,\bar{Y})}\}$$
$$-i\{\overline{N(\bar{X},\bar{Y})}-\overline{N(X,Y)}+\overline{\overline{N(\bar{X},Y)}}+\overline{\overline{N(X,\bar{Y})}}\}.$$

Substituting from (4.23) in (4.22), we obtain (4.20)a. (4.20)b can similarly be obtained.

LEMMA (4.1)b. *We also have*

(4.24)a $$(dL)(m(X),m(Y))=-\overset{a}{A}(X)\overset{b}{A}(Y)\overset{x}{p}([T_a,T_b])p_x$$

(4.24)b $$(dM)(m(X),m(Y))=-\overset{a}{A}(X)\overset{b}{A}(Y)\overset{z}{q}([T_a,T_b])Q_z$$

PROOF. In consequence of (4.22), (4.13) and (4.4) we have

$$(dL)(m(X),m(Y))=-L([m(X),m(Y)])$$
$$=-L(\overset{a}{A}(X)\overset{b}{A}(Y)[T_a,T_b]+\overset{a}{A}(X)T_a(\overset{b}{A}(Y))T_b$$
$$-\overset{b}{A}(Y)T_b(\overset{a}{A}(X))T_a=-\overset{a}{A}(X)\overset{b}{A}(Y)\overset{z}{p}([T_a,T_b])p_z$$

We similarly have (4.24)b.

LEMMA (4.1)c. *We also have*

(4.25)a $$2(dL)(m(X),m(Y))=\overline{\overline{[m(X),m(Y)]}}+i\,\overline{[m(X),m(Y)]},$$

(4.25)b $$2(dM)(m(X), m(Y)) = [\overline{\overline{m(X), m(Y)}}] - i[\overline{m(X), m(Y)}].$$

The proof is obvious.

LEMMA (4.2) *We have*

(4.26)a $$2(dL)(M(X), M(Y)) = [\overline{\overline{M(X)}, \overline{M(Y)}}] + i[\overline{M(X), M(Y)}],$$

(4.26)b $$2(dM)(L(X), L(Y)) = \overline{[\overline{L(X)}, \overline{L(Y)}]} - i\overline{[L(X), L(Y)]},$$

(4.27)a $$2(dL)(M(X), M(Y)) = \overset{x}{q}(X)\overset{y}{q}(Y)\{\overline{\overline{[\underset{x}{Q}, \underset{y}{Q}]}} + i\overline{[\underset{x}{Q}, \underset{y}{Q}]}\},$$

(4.27)b $$2(dM)(L(X), L(Y)) = \overset{x}{p}(X)\overset{y}{p}(Y)\{\overline{\overline{[\underset{x}{p}, \underset{y}{p}]}} - i\overline{[\underset{x}{p}, \underset{y}{p}]}\},$$

(4.28)a $$4(dL)(M(X), M(Y)) = \overline{\overline{N(\overline{X}, \overline{Y})}} + i\overline{N(\overline{X}, \overline{Y})},$$

(4.28)b $$4(dM)(L(X), L(Y)) = \overline{\overline{N(\overline{X}, \overline{Y})}} - i\overline{N(\overline{X}, \overline{Y})}.$$

The proof follows the pattern of the proof of Lemma (4.1).

LEMMA (4.3). *The following equations are satisfied*

(4.29)a $$(dm)(L(X), L(Y)) = -\overset{x}{A}([L(X), L(Y)])\underset{x}{T},$$

(4.29)b $$(dm)(M(X), M(Y)) = -\overset{x}{A}([M(X), M(Y)])\underset{x}{T},$$

(4.30)a $$(dm)(L(X), L(Y)) = -\overset{x}{p}(X)\overset{y}{p}(Y)\overset{z}{A}([\underset{x}{p}, \underset{y}{p}])\underset{z}{T},$$

(4.30)b $$(dm)(M(X), M(Y)) = -\overset{x}{q}(X)\overset{y}{q}(Y)\overset{z}{A}([\underset{x}{Q}, \underset{y}{Q}])\underset{z}{T},$$

(4.31)a $$(dm)(L(X), L(Y)) = \overset{z}{A}(N(X, Y))\underset{z}{T} - \overset{z}{A}(N(\overline{X}, \overline{Y}))\underset{z}{T}$$
$$+ i\{\overset{z}{A}(N(\overline{X}, Y))\underset{z}{T} + \overset{z}{A}(N(X, \overline{Y}))\underset{z}{T}\},$$

(4.31)b $$(dm)(M(X), M(Y)) = \overset{z}{A}(N(X, Y) - N(\overline{X}, \overline{Y}))\underset{z}{T}$$
$$+ i\{\overset{z}{A}(N(\overline{X}, Y) + N(X, \overline{Y}))\underset{z}{T}\}$$

The proof follows the pattern of the proof of Lemma (4.1).

THEOREM (4.4). *In order that* Π_{n-2m} *be integrable, it is necessary and sufficient that*

(4.32)a $$\overline{\overline{N(\overline{X}, \overline{Y})}} - \overline{\overline{N(X, Y)}} - \overline{N(\overline{X}, Y)} - \overline{N(X, \overline{Y})} = 0,$$

equivalent to

(4.32)b $$\overline{N(\overline{X}, \overline{Y})} - \overline{N(X, Y)} + \overline{\overline{N(\overline{X}, Y)}} + \overline{\overline{N(X, \overline{Y})}} = 0.$$

or

(4.33) $$\overset{a}{A}(X)\overset{b}{A}(Y)\overset{x}{p}([\underset{a}{T},\underset{b}{T}])\underset{x}{p}=\overset{a}{A}(X)\overset{b}{A}(Y)\overset{z}{q}([\underset{a}{T},\underset{b}{T}])\underset{z}{Q}=0,$$

or

(4.34)a $$\overline{[m(X),m(Y)]}=0,$$

equivalent to

(4.34)b $$\overline{\overline{[m(X),m(Y)]}}=0$$

PROOF. The distribution Π_{n-2m} is given by

(4.35) a) $L(X)=0$, b) $M(X)=0$, c) $X=m(X)$.

In order that Π_{n-2m} be completely integrable it is necessary and sufficient that $L(X)=0$, $M(X)=0$ be completely integrable, that is

(4.36) a) $(dL)(X,Y)=0$, b) $(dM)(X,Y)=0$,

be satisfied by any vector satisfying (4.35)c. Substituting from (4.35)c into (4.36)a, b we get

(4.37) a) $(dL)(m(X),m(Y))=0$, b) $(dM)(m(X),m(Y))=0$.

Using (4.37) in (4.20), (4.24), (4.25), we obtain (4.32), (4.33) and (4.34) respectively.

THEOREM (4.5). *In order that Π_m and $\bar{\Pi}_m$ are completely integrable it is necessary and sufficient that*

(4.38)a $$\overline{\overline{[L(X),L(Y)]}}=i\,\overline{[L(X),L(Y)]},$$

(4.38)b $$\overset{a}{A}(L(X),L(Y))=0,$$

(4.38)c $$\overline{\overline{[M(X),M(Y)]}}+i\,\overline{[M(X),M(Y)]}=0,$$

(4.38)d $$\overset{a}{A}(M(X),M(Y))=0;$$

or

(4.39)a $$\overset{x}{p}(X)\overset{y}{p}(Y)\{\overline{\overline{[\underset{x}{p},\underset{y}{p}]}}-i\,\overline{[\underset{x}{p},\underset{y}{p}]}\}=0,$$

(4.39)b $$\overset{x}{p}(X)\overset{y}{p}(Y)\overset{a}{A}([\underset{x}{P},\underset{y}{P}])=0,$$

(4.39)c $$\overset{x}{q}(X)\overset{y}{q}(Y)\{\overline{\overline{[\underset{x}{Q},\underset{y}{Q}]}}+i\,\overline{[\underset{x}{Q},\underset{y}{Q}]}\}=0$$

(4.39)d $$\overset{x}{q}(X)\overset{y}{q}(Y)\overset{a}{A}([\underset{x}{Q},\underset{y}{Q}])=0$$

or

(4.40)a $$\overline{N(\bar{X},\bar{Y})}=0$$

(4.40)b $$\overset{a}{A}(N(X,Y))=\overset{a}{A}(N(\bar{X},\bar{Y})).$$

PPOOF. The distribution Π_m is given by

(4.41) a) $M(X)=0$, b) $m(X)=0$, c) $X=L(X)$.

In order that Π_m be completely integrable, it is necessary and sufficient that $M=0$ and $m=0$ be completely integrable, that is

(4.42) a) $(dM)(X,Y)=0$, b) $(dm)(X,Y)=0$,

be satisfied by any vector satisfying (4.41)c. Substituting from (4.41)c in (4.42) a, b, we get

(4.43) a) $(dM)(L(X),L(Y))=0$, b) $(dm)(L(X),L(Y))=0$.

By virtue of (4.43)a, b, the equations (4.26)b and (4.29)a, assume the forms (4.38)a, b. Similarly (4.38)c, d are the necessary and sufficient conditions for the integrability of $\tilde{\Pi}_m$. It can similarly be proved that (4.39)a, b are the necessary and sufficient conditions for the integrability of Π_m and (4.39)c, d are the necessary and sufficient conditions for the integrability of $\tilde{\Pi}_m$.

From (4.43)a and (4.28)b, we have

(4.44)a $$\overline{\overline{N(\overline{X},\overline{Y})}}-i\overline{N(\overline{X},\overline{Y})}=0.$$

From (4.43)b and (4.31)a, we have

(4.44)b $$\{\overset{a}{A}(N(X,Y)-N(\overline{X},\overline{Y}))\}\underset{a}{T}+i\{\overset{a}{A}(N(\overline{X},Y)+N(X,\overline{Y}))\}\underset{a}{T}=0.$$

We can similarly have when we consider the integrability of $\tilde{\Pi}_m$

(4.45)a $$\overline{\overline{N(\overline{X},\overline{Y})}}+i\overline{N(\overline{X},\overline{Y})}=0,$$

(4.45)b $$[\overset{a}{A}(N(X,Y)-N(\overline{X},\overline{Y}))-i\{\overset{a}{A}(N(\overline{X},Y)+N(X,\overline{Y}))\}]\underset{a}{T}=0.$$

Thus from (4.45)a and (4.45)b the necessary and sufficient condition for the integrability of Π_m ($\tilde{\Pi}_m$) are (4.40).

THEOREM (4.6). *In order that V_n be completely integrable, it is necessary and sufficient that*

(4.46)a $$\overline{N(\overline{X},\overline{Y})}=0,$$

(4.46)b $$\overset{a}{A}(N(X,Y))=\overset{a}{A}(N(\overline{X},\overline{Y}))=0.$$

PROOF. (4.46) follows from (4.40) and (4.32).

THEOREM (4.7). *The necessary and sufficient condition that a differentiable manifold V_n with an f-structure be integrable is that it is possible to introduce an affine connexion D with respect to which f is covariant constant and which is such* *that*

(4.47)a $$\overline{S(\overline{X},\overline{Y})}=0.$$

(4.47)b $$\overset{a}{A}(S(X,Y))=\overset{a}{A}(S(\overline{X},\overline{Y})).$$

where S is the torsion tensor of D.

PROOF. Let B be a symmetric connexion in V_n and D another connexion. Let

(4.48) $$D_XY=B_XY+H(X,Y).$$

Let us assume

(4.49)a $$(D_Xf)(Y)=0$$

equivalent to

(4.49)b $$D_X\overline{Y}=\overline{D_XY}$$

Using (4.48) in (4.49)b, we get

$$\overline{B_XY}-B_X\overline{Y}=H(X,\overline{Y})-\overline{H(X,Y)},$$

whence

(4.50) $$\overline{\overline{H(\overline{X},\overline{\overline{Y}})}}+\overline{H(\overline{X},\overline{Y})}=-\overline{B_X\overline{Y}}-\overline{\overline{B_{\overline{X}}\overline{\overline{Y}}}}.$$

Since by barring Y and the whole equation, we again have the same equation, we shall attempt a solution of this equation. The general solution of this equation is given by

(4.51)a $$\overline{4H(\overline{X},\overline{Y})}=-\overline{2B_{\overline{X}}\overline{Y}}-\overline{\overline{2B_{\overline{X}}\overline{\overline{Y}}}}+\overline{W(X,\overline{Y})}-\overline{\overline{W(\overline{X},\overline{\overline{Y}})}},$$

where W is an arbitrary vector valued bilinear function. For,

(4.51)b $$\overline{\overline{4H(\overline{X},\overline{\overline{Y}})}}=-\overline{\overline{2B_{\overline{X}}\overline{\overline{Y}}}}-\overline{2B_{\overline{X}}\overline{Y}}+\overline{\overline{W(\overline{X},\overline{\overline{Y}})}}-\overline{W(\overline{X},\overline{Y})},$$

whence by adding the last two equations, we obtain (4.50).

Let us put

$$\overline{2W(\overline{X},\overline{Y})}=-\overline{\overline{B_{\overline{Y}}\overline{\overline{X}}}}-\overline{B_{\overline{Y}}\overline{X}}-\overline{B_{\overline{\overline{Y}}}\overline{\overline{X}}}+\overline{\overline{B_{\overline{\overline{Y}}}\overline{X}}}.$$

Then

$$\overline{\overline{2W(\overline{X},\overline{\overline{Y}})}}=\overline{B_{\overline{\overline{Y}}}\overline{\overline{X}}}-\overline{\overline{B_{\overline{\overline{Y}}}\overline{X}}}+\overline{\overline{B_{\overline{Y}}\overline{\overline{X}}}}+\overline{B_{\overline{Y}}\overline{X}}.$$

Consequently (4.51)a can be written as

$$\overline{4H(\overline{X},\overline{Y})}=-\overline{2B_{\overline{X}}\overline{Y}}-\overline{\overline{2B_{\overline{X}}\overline{\overline{Y}}}}-\overline{B_{\overline{\overline{Y}}}\overline{\overline{X}}}-\overline{\overline{B_{\overline{Y}}\overline{X}}}-\overline{\overline{B_{\overline{Y}}\overline{\overline{X}}}}-\overline{B_{\overline{Y}}\overline{X}},$$

From this equation we have

$$\overline{4S(\bar{X},\bar{Y})} = -\overline{B_{\bar{X}}\bar{Y}} + \overline{B_{\bar{Y}}\bar{X}} - \overline{\overline{B_{\bar{X}}\bar{\bar{\bar{Y}}}}} + \overline{\overline{B_{\bar{Y}}\bar{\bar{\bar{X}}}}} - \overline{\overline{B_{\bar{\bar{Y}}}\bar{\bar{\bar{X}}}}} + \overline{\overline{B_{\bar{\bar{X}}}\bar{\bar{\bar{Y}}}}} + \overline{\overline{B_{\bar{\bar{Y}}}\bar{X}}} - \overline{\overline{B_{\bar{\bar{X}}}\bar{Y}}}$$

$$= \overline{N(\bar{X},\bar{Y})}.$$

If V_n is integrable $\overline{N(X,Y)}=0$. Consequently we have (4.47)a.

From (4.48) and (4.49), we have

$$\overset{a}{A}(H(\bar{X},\bar{Y})) = -\overset{a}{A}(B_{\bar{X}}\bar{Y}),$$

$$\overset{a}{A}(H(\bar{\bar{X}},\bar{\bar{Y}})) = -\overset{a}{A}(B_{\bar{\bar{X}}}\bar{\bar{Y}}).$$

Consequently

$$\overset{a}{A}(-S(\bar{X},\bar{Y})+S(\bar{\bar{X}},\bar{\bar{Y}})) = \overset{a}{A}(B_{\bar{X}}\bar{Y}-B_{\bar{Y}}\bar{X}) - \overset{a}{A}(B_{\bar{\bar{X}}}\bar{\bar{Y}}-B_{\bar{\bar{Y}}}\bar{\bar{X}}).$$

$$= \overset{a}{A}(N(X,Y)) - \overset{a}{A}(N(\bar{X},\bar{Y})).$$

If V_n is integrable $\overset{a}{A}(N(X,Y))=0$ and we have (4.47)b.

Ishihara and Yano (1964) obtained the integrability conditions of Π_{2m} and Π_{n-2m} as

$$m(N(X,Y))=0,$$

and

$$N(m(X),m(Y))=0,$$

respectively. We will obtain these in other forms. It can be proved on the lines of the proof of Theorem (4.5), that the necessary and sufficient conditions that Π_m be integrable is that

$$m([l(X),\ l(Y)])=0,$$

equivalent to

$$\overline{\overline{[\bar{\bar{X}},\bar{\bar{Y}}]}} + [\bar{\bar{X}},\bar{\bar{Y}}]=0.$$

or

$$\overline{\overline{[\bar{X},\bar{Y}]}} + [\bar{X},\bar{Y}]=0.$$

This equation is equivalent to

$$N(X,Y)+\overline{\overline{N(X,Y)}}=0. \tag{4.52}$$

Similarly the necessary and sufficient condition that Π_{n-2m} be integrable is

$$l([m(X),m(Y)]=0,$$

which is equivalent to

$$\overline{\overline{[\bar{\bar{X}},\bar{\bar{Y}}]}} + \overline{\overline{[\bar{\bar{X}},Y]}} + \overline{\overline{[X,\bar{\bar{Y}}]}} + \overline{\overline{[X,Y]}}=0, \tag{4.53}$$

or

$$\overline{\overline{N(\bar{X},\bar{Y})}}-\overline{\overline{N(X,Y)}}-\overline{N(\bar{X},Y)}-\overline{N(X,\bar{\bar{Y}})}=0,$$

or

(4.54)a $$\overline{N(\bar{X},\bar{Y})}-\overline{N(X,Y)}+\overline{\overline{N(\bar{X},Y)}}+\overline{\overline{N(X,\bar{Y})}}=0.$$

From (4.52) and (4.54)a it is clear that the necessary and sufficient condition that V_n be integrable is

$$N(X,Y)=0.$$

a condition obtained by Ishihara and Yano (1964).

It is interesting to note that (4.53) is also equivalent to

(4.54)b $$N(\bar{\bar{X}},\bar{\bar{Y}})+N(X,Y)+N(\bar{\bar{X}},Y)+N(X,\bar{\bar{Y}})=0.$$

From (4.52) and (4.54)b, we again get the same necessary and sufficient condition for the integratibly of V_n.

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