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FLOW BETWEEN PARALLEL DISKS

By Satya Prakash

Introduction.

The effects of porous boundaries on steady laminar flow have been studied for different geometries. The flow in a straight channel with porous walls has been considered by Berman [1], Proudman [2], Terrill [3], Sellar [4], Yuan [5], and several others. Berman [6], has also considered the flow in a porous circular pipe and in a porous annulus. Prakash [7] has considered the flow along a corner bounded by two porous parabolic walls and the flow through a channel bounded by two confocal porous parabolic walls. Rasmussen [8] has considered the flow between two parallel porous equal coaxial disks when the inflow or the outflow at the disks are equal. He has studied the flow near the axis of symmetry on the assumption that the distance between the disks is very small compared to their radius. In the present paper, the author considers the flow between two disks when the inflow or the outflow at the porous disk is prescribed and the other disk is not porous. Here also the flow near the axis of symmetry has been studied on the assumption that the distance between the disks is very small compared to their radius. It has not been possible to obtain explicit solutions of the above problem for all values of the Reynolds number. Therefore attempt has been made to obtain approximate solutions of the problem for specialized values of the Reynolds number. Solutions of the problem for small Reynolds number have been obtained by the use of an asymptotic series. And then some conclusion have been derived.

Formulation of the Problem.

Consider steady incompressible laminar viscous flow between two parallel coaxial stationary circular disks of radius R distant L apart when the inflow or the outflow at the porous disk is prescribed and the other disk is not porous. Suppose that L is very small in comparison with R. Take z-axis along the axis of symmetry and x, y-axes along two mutually perpendicular lines in a plane perpendicular to the z-axis. Let r, θ, z denote the radial, azimuthal and axial coordinates of a point in the region of flow. Let u and v be the components of the fluid velocity in the axial and radial directions respectively. And assume that the component of the fluid velocity in the azimuthal direction is zero. The boundary conditions for the problem under consideration are

$$u=0 \text{ at } z=0 \text{ and } u=\mp 2v \text{ at } z=L \text{ for } 0<r<R \tag{1}$$

$$v=0 \text{ at } z=0 \text{ and } v=0 \text{ at } z=L \text{ for } 0<r<R \tag{2}$$

It follows in view of the conditions (1) and (2) that u and v will be independent of θ. Hence the equation of continuity and the equations of motion in the present case reduce to

$$\frac{1}{r}\frac{\partial}{\partial r}(rv)+\frac{\partial u}{\partial z}=0 \tag{3}$$

$$\left.\begin{aligned} v\frac{\partial v}{\partial r}+u\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial z^2}-\frac{v}{r^2}\right) \\ 0 &= -\frac{1}{\rho}\frac{\partial p}{\partial \theta} \\ v\frac{\partial u}{\partial r}+u\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}\right) \end{aligned}\right\} \tag{4}$$

Here ρ is the fluid density, ν is the coefficient of kinematic viscosity and p is the fluid pressure.

As L is very small in comparison with R, it is reasonable to expect that u depends only on z for $r \ll R$ and therefore we may take

$$u=2F(z) \tag{5}$$

Substituting this value of u in eqn. (3), we get

$$\frac{1}{r}\frac{\partial}{\partial r}(rv)+2F'(z)=0$$

Integrating, we get

$$v=-rF'(z)+c/r, \tag{6}$$

where c is an arbitrary constant.

As v should be finite at $r=0$, we must have $c=0$. Hence eqn. (6) gives

$$v=-rF'(z) \tag{7}$$

Substituting the values of u and v given by eqns. (5) and (7) in the third of the eqns. (4), we obtain

$$4F(z)F'(z)=-\frac{1}{\rho}\frac{\partial p}{\partial z}+2\nu F''(z)$$

Integrating and noting that p is independent of θ as is seen from the second of the eqns. (4), we have

$$2F^2(z)=-\frac{p}{\rho}+2\nu F'(z)+\phi(r), \tag{8}$$

where $\phi(r)$ is an arbitrary function of r.

Substituting the values of u, v and p given by eqns. (5), (7), (8) in the first of the eqns (4), we get after simplification,

$$2F(z)F''(z)-F'^2(z)-\nu F'''(z)=\frac{1}{r}\frac{d\phi}{dr} \tag{9}$$

Now as the left-hand side of eqn. (9) is a function of z only and its right-hand side is a function of r only it follows that each side must be equal to a constant quantity and therefore we have

$$2F(z)F''(z)-F'^2(z)-\nu F'''(z)=k. \tag{10}$$

where k is a constant quantity.

Differentiating eqn. (10) with respect to z, we have

$$2F(z)F'''(z)-\nu F''''(z)=0 \tag{11}$$

From eqns. (1) and (2), we see that boundary conditions on $F(z)$ become

$$F(z)=0 \text{ at } z=0 \text{ and } F(z)=\mp V \text{ at } z=L \tag{12}$$

$$F'(z)=0 \text{ at } z=0 \text{ and } F'(z)=0 \text{ at } z=L \tag{13}$$

Now we shall rewrite the eqns. (11), (12), (13) in dimensionless form. For this, we take a characteristic length L and a characteristic velocity V. Introducing

$$f(\xi)=\frac{F(z)}{V} \text{ and } \xi=\frac{z}{L}, \tag{14}$$

the eqns. (11), (12), (13) become

$$2f(\xi)f'''(\xi)-\frac{1}{R}f''''(\xi)=0, \tag{15}$$

where $R=VL/\nu$ is the Reynolds number.

$$f(\xi)=0 \text{ at } \xi=0 \text{ and } f(\xi)=\mp 1 \text{ at } \xi=1 \tag{16}$$

$$f'(\xi)=0 \text{ at } \xi=0 \text{ and } f'(\xi)=0 \text{ at } \xi=1 \tag{17}$$

Since eqn. (15) is nonlinear, it is not possible to obtain its explicit solutions satisfying conditions (16) and (17) for all values of the Reynolds number. We shall obtain approximate solutions of this problem for the cases when the porous disk is subjected to injection and when it is subjected to suction on the assumption that the Reynolds number is small.

Porous Disk with Injection.

When the porous disk is subjected to injection, the boundary value problem given by eqns. (15), (16), (17) undergoes a slight change in the sense that the condition $f(\xi)=1$ at $\xi=1$ is ruled out. In this case, $f(\xi)$ may be assumed to take the form

$$f(\xi)=f_0(\xi)+Rf_1(\xi)+R^2f_2(\xi)+\cdots\cdots\cdots \tag{18}$$

Substituting this value of $f(\xi)$ in eqn. (15), we get after rearranging the terms,

$$f_0''''(\xi)+[f_1''''(\xi)-2f_0(\xi)f_0'''(\xi)]R$$
$$+[f_2''''(\xi)-2f_0(\xi)f_1'''(\xi)-2f_0'''(\xi)f_1(\xi)]R^2+\cdots\cdots=0 \qquad (19)$$

Hence setting coefficients of various powers of R separately equal to zero, we obtain

$$f_0''''(\xi)=0 \qquad (20)$$
$$f_1''''(\xi)-2f_0(\xi)f_0''''(\xi)=0 \qquad (21)$$
$$f_2''''(\xi)-2f_0(\xi)f_1'''(\xi)-2f_0'''(\xi)f_1(\xi)=0 \qquad (22)$$

and so on.

The boundary conditions associated with eqns. (20), (21), (22) are given by

$$f_0(0)=0,\ f_0(1)=-1,\ f_0'(0)=0,\ f_0'(1)=0 \qquad (23)$$
$$f_1(0)=0,\ f_1(1)=0,\ f_1'(0)=0,\ f_1'(1)=0 \qquad (24)$$
$$f_2(0)=0,\ f_2(1)=0,\ f_2'(0)=0,\ f_2'(1)=0 \qquad (25)$$

The eqns. (23), (24), (25) have been obtained by substituting the value of $f(\xi)$ given by (18) in eqns. (16) and (17) and then equating coefficients of various powers of R on both sides of the resulting equations.

The solution of eqn. (20) satisfying the boundary conditions (23) is given by

$$f_0(\xi)=-3\xi^2+2\xi^3 \qquad (26)$$

The solution of eqn. (21) satisfying the boundary conditions (24) is given by

$$f_1(\xi)=-\frac{13}{35}\xi^2+\frac{18}{35}\xi^3-\frac{1}{5}\xi^6+\frac{2}{35}\xi^7 \qquad (27)$$

And the solution of eqn. (22) satisfying the boundary conditions (25) is given by

$$f_2(\xi)=-\frac{624}{40425}\xi^2+\frac{23}{539}\xi^3-\frac{8}{105}\xi^6+\frac{36}{1225}\xi^7$$
$$+\frac{1}{21}\xi^9-\frac{6}{175}\xi^{10}+\frac{12}{1925}\xi^{11} \qquad (28)$$

Hence the solution of eqn. (15) satisfying the boundary conditions (16) and (17) when the condition $f(\xi)=1$ at $\xi=1$ has been ruled out is given by

$$f(\xi)=-3\xi^2+2\xi^3+\left(-\frac{13}{35}\xi^2+\frac{18}{35}\xi^3-\frac{1}{5}\xi^6+\frac{2}{35}\xi^7\right)R+\left(-\frac{624}{40425}\xi^2+\frac{23}{539}\xi^3\right.$$
$$\left.-\frac{8}{105}\xi^6+\frac{36}{1225}\xi^7+\frac{1}{21}\xi^9-\frac{6}{175}\xi^{10}+\frac{12}{1925}\xi^{11}\right)+O(R^3) \qquad (29)$$

Differentiating both sides of eqn. (29) with respect to ξ, we obtain

$$f'(\xi)=-6\xi+6\xi^2+\left(-\frac{26}{35}\xi+\frac{54}{35}\xi^2-\frac{6}{5}\xi^5+\frac{2}{5}\xi^6\right)R+\left(-\frac{1248}{40425}\xi+\frac{69}{539}\xi^2\right.$$

$$-\frac{16}{35}\xi^5+\frac{36}{175}\xi^6+\frac{3}{7}\xi^8-\frac{12}{35}\xi^9+\frac{12}{175}\xi^{10}\Big)R^2+O(R^3) \tag{30}$$

Hence with the aid of eqns. (5), (7), (14), (29), (30), we have

$$\frac{u}{V}\equiv 2f(\xi)=-6\xi^2+4\xi^3+\Big(-\frac{26}{35}\xi^2+\frac{36}{35}\xi^3-\frac{2}{5}\xi^6+\frac{4}{35}\xi^7\Big)R$$
$$+\Big(-\frac{1248}{40425}\xi^2+\frac{46}{539}\xi^3-\frac{16}{105}\xi^6+\frac{72}{1225}\xi^7+\frac{2}{21}\xi^9$$
$$-\frac{12}{175}\xi^{10}+\frac{24}{1925}\xi^{11}\Big)R^2+O(R^3) \tag{31}$$

and

$$\frac{1}{(r/L)}\Big(\frac{v}{V}\Big)\equiv -f'(\xi)=6\xi-6\xi^2+\Big(\frac{26}{35}\xi-\frac{54}{35}\xi^2+\frac{6}{5}\xi^5-\frac{2}{5}\xi^6\Big)R$$
$$+\Big(\frac{1248}{40425}\xi-\frac{69}{539}\xi^2+\frac{16}{35}\xi^5-\frac{36}{175}\xi^6-\frac{3}{7}\xi^8$$
$$+\frac{12}{35}\xi^9-\frac{12}{175}\xi^{10}\Big)R^2+O(R^3) \tag{32}$$

Porous Disk with Suction.

When the porous disk is subjected to suction, the boundary value problem given by eqns. (15), (16), (17) would be deprived of the condition $f(\xi)=-1$ at $\xi=1$. Proceeding in a similar way as in the previous case, we finally obtain

$$\frac{v}{V}\equiv 2f(\xi)=6\xi^2-4\xi^3+\Big(-\frac{26}{35}\xi^2+\frac{36}{35}\xi^3-\frac{2}{5}\xi^6+\frac{4}{35}\xi^7\Big)R$$
$$+\Big(\frac{1248}{40425}\xi^2-\frac{46}{539}\xi^3+\frac{16}{105}\xi^6-\frac{72}{1225}\xi^7-\frac{2}{21}\xi^9$$
$$+\frac{12}{175}\xi^{10}-\frac{24}{1925}\xi^{11}\Big)R^2+O(R^3) \tag{33}$$

and

$$\frac{1}{(r/L)}\Big(\frac{v}{V}\Big)\equiv -f'(\xi)=-6\xi+6\xi^2+\Big(\frac{26}{35}\xi-\frac{54}{35}\xi^2+\frac{6}{5}\xi^5-\frac{2}{5}\xi^6\Big)R$$
$$+\Big(-\frac{1248}{40425}\xi+\frac{69}{539}\xi^2-\frac{16}{35}\xi^5+\frac{36}{175}\xi^6+\frac{3}{7}\xi^8$$
$$-\frac{12}{35}\xi^9+\frac{12}{175}\xi^{10}\Big)R^2+O(R^3) \tag{34}$$

Numerical Results and Conclusions.

Eqns. (31), (32), (33), (34) give the solutions of the proposed problem in dimensionless form for the two respective cases in which the porous disk is subjected to injection and suction separately. Table I depicts the values of $\frac{u}{V}=2f(\xi)$ and

$\frac{1}{(r/L)}\left(\frac{v}{V}\right)=-f'(\xi)$ against ξ in both the cases with $R=.01$.

Table I

Values of u/V and $[1/(r/1)]\,v/V$ against ξ with $R=.01$.

ξ	u/V(injection)	$[1/(r/1)]\,v/V$(injection)	u/V(suction)	$[1/(r/1)]\,v/V$ (suction)
.00	.0000	.0000	.0000	.0000
.10	−.0560	.5405	.0559	−.5394
.20	−.2082	.9608	.2077	−.9591
.30	−.4323	1.2608	.4316	−1.2591
.40	−.7045	1.4406	.7034	−1.4393
.50	−1.0006	1.5001	.9993	−1.4998
.60	−1.2966	1.4396	1.2953	−1.4403
.70	−1.5684	1.2591	1.5675	−1.2608
.80	−1.7922	.9589	1.7917	−.9610
.90	−1.9446	.5391	1.9439	−.5408
1.00	−2.0000	.0000	2.0000	.0000

From the above table, we see that the velocity distributions in the two cases are quite different in their basic structures, a fact easily seen directly from eqns. (31), (32), (33), (34) also. In the case of injection, the radial velocity is nonnegative meaning thereby that liquid flows radialy away from the axis of symmetry; and in the case of suction it is nonpositive meaning thereby that liquid flows radially inwards. It is also displayed that the radial velocity in the two cases differs in magnitude too. Likewise it is displayed that the axial velocity in the two cases differs in magnitude as well as in direction. It may be pointed out that the reversal of the direction of the velocities in the two cases is a natural consequence of the fact that in one case the porous disk is subjected to injection and in the other case it is subjected to equal suction; however variations in magnitude of the velocities in the two cases fail to admit such simple explantions based on physical intuition. Finally it may be obseved that the numerically greatest value of the radial velocity in both the cases is attained at $\xi=5$, which is midway from the two disks.

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Birla Institute of Technology and Science,
Pilani, Rajasthan, India

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