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BERS' DUALITY OF AUTOMORPHIC FORMS OF EXTENDED KLEINIAN GROUP

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Introduction.

Bers proved that the Petorson scalar product establishes a topological isomorphism between Automorphic forms $A_q^p(D,G)$ and the linear functional on $A_q^{p'}(D,G)$ of a Kleinian group G. In this paper we prove that the isomorphism exists also between the automorphic forms of an extended Kleinian group.

Let E be a group of transformations $\dfrac{az+b}{az+d}$ or $\dfrac{a\bar{z}+b}{c\bar{z}+d}$ with $ad-bc=1$. If G is the set of all Möbius transformations contained in E then G forms a normal subgroup of E with index two, and E acts on the extended complex plane $\mathbf{C}$.

If the region of discontinuity D of E is not empty, then we call E an *extended (Kleinian) group.*

Let f be a complex valued function defined on an open set in the complex plane and if the conjugate function $\bar{f}$ is analytic then we call f an *anti-analytic function.*

The orbit space D/E forms a surface with boundary.

Let f be analytic or anti-analytic mapping of D into the extended complex plane. Let r and s be integers. Then for every function h on $f(D)$, we define

$$(f^*{}_r h)(z)=h(f(z))\ f'(z)^r$$

for analytic f, and

$$(f^*{}_r h)(z)=\overline{h(f(z))f'(z)^r}$$

where f' denotes $\dfrac{\partial f}{\partial \bar{z}}$, for anti-analytic f.

If the region of discontinuity of E does not contain at least three points of $C\cup\{\infty\}$, then we call E a (non-elementary) extended group.

Let E be non-elementary and D be an invariant union of components of the region of discontinuity of E.

Let D be the Poincare metric (the unique complete conformal Riemannian metric defined on each component of D with constant curvature -4).

Let f be analytic or anti-analytic mapping of D into the extended complex plane. Then for every function h on $f(D)$, we define $f^*{}_r h$ on D by

$$(f^*{}_r h)(z)=h(f(z))f'(z)^r$$

for analytic f, and

$$(f^*{}_r h)(z)=\overline{h(f(z))f'(z)}^r$$

for anti-analytic f and f' denotes $\frac{\partial f}{\partial \bar{z}}$

Let $q\geq 2$ be a fixed integers. A measurable automorphic form of weight $(-2q)$ is a measurable function f that satisfies $A^*{}_q f=f$, for all $A\in E$.

For each p with $1\leq p\leq\infty$, the measurable automorphic forms with

$$\|f\|_{q,\infty}=\sup_{z\in D}\ \lambda(z)^{-q}|f(z)|<\infty,\ \text{for } p=\infty$$

$$(\text{or } \|f\|^p_{q,p\ D/E}=\iint_{D/E}\lambda(z)^{2-qp}|f(z)|^p|dz\wedge d\bar{z}|<\infty)$$

form a Banach space $L_q^\infty(D,E)$ (or $L_q^p(D,E)$) of *bounded forms* (or *p-integrable forms*).

A *holomorphic form* is a holomorphic measurable automorphic form which satisfies

$$\lim_{z\to\infty} f(z)=0\ (|z|^{-2q})\ \text{if } \infty\in D,$$

For $p\geq 1$, the holomorphic forms in $L_q^p(D,E)$ form a closed subspace denoted by $A_q^p(D,E)$.

Let D be an open set in the extended complex plane with at least three boundary points and f be a conformal or anticonformal mapping of D into the extended complex plane. Let U be the unit disc, define

$$K_U(z,t)=\frac{1}{2}(2q^{-1})\pi^{q-1}\ k_U(z,t)^q,$$

$$k_U(z,t)=\pi^{-1}(1-z\bar{t})^{-2}.$$

Let h: $U\to D$ be a one to one conformal mapping then define,

$$K_D(h(z),\ h(t))h'(z)\ \overline{h'(t)}=k_U(z,t),$$

$$K_D(z,t)=\frac{1}{2}(2q-1)\ \pi^{q-1}iK_D(z,t)^q.$$

For a measurable function f on D define

$$(\beta_q f)(z)=\iint_D\lambda(t)^{2-2q}K_D(z,t)f(t)\ dt\wedge d\bar{t},\ \ z\in D$$

whenever the integral is absolutely convergent.

LEMMA 1. *The operator β_q is a bounded real linear projection of $L_q^p(D, E)$ onto $A_q^p(D, E)$.*

PROOF. We know this is true for Kleinian group G, and hence β_q maps $L_q^p(D, E)$ into $A_q^p(D, G)$. If f is anticonformal then $K_{f(D)}(f(z), f(t)) f'(z)^q\overline{f'(t)^q} = \overline{-K_D(z, t)}$. By the above identity we know that β_q maps $L_q^p(D, E)$ onto $A_q^p(D, E)$. If $f \in L_q^p(D, G)$ and $g \in L_q^{p'}(D, G)$ with $\frac{1}{p}+\frac{1}{p'}=1$,

we define

$$(f, g)_{q,G} = \iint_{D/G} \lambda(z)^{2-2q} f(z)\overline{g(z)} dz \wedge d\bar{z},$$

then we have

(1) $$(\beta_q f, g)_{q,G} = (f, \beta_q g)_{q,G}.$$

If $f \in L_q^p(D, E)$ and $g \in L_q^{p'}(D, E)$, we define

$$(f, g)_{q,X} = \iint_X \lambda(z)^{2-2q} f(z)\overline{g(z)} dz \wedge d\bar{z}$$

where X is a fixed fundamental domain of E. We have

(2) $$(f, g)_{q,X} = -\overline{(f, g)}_{q,U(X)}$$

where U is an anti-analytic element in E. (1) is well known, see [5], and (2) is a simple calculation.

For f holomorphic on D, define the *Poincaré series* of f by

$$(\theta_q f)(z) = \sum_{A \in E} (A^*{}_q f)(z), \ z \in D$$

whenever the right side converges abs lutely and uniformly on compact subsets of D.

THEOREM 1. *For an open set D (with at least three boundary points) the mapping β_q is a continuous linear mapping of $A_q^1(D)$ onto $A_q^1(D, E)$, with norm less than* 1. *Furthermore, for every g in $A_q^p(D, E)$ there is an f in $A_q^p(D)$ such that*

(3) $$g = \theta_q f$$

PROOF. For f in $A_q^1(D)$, the convergence and holomorphicity of $\theta_q(f)$ can be proven exactly as the Kleinian group case, hence we do not prove it here. Let G be the maximal Kleinian subgroup contained in E. Let U and B be two anti-analytic elements in E. We calculate

$$\begin{aligned}
&\overline{(\theta_q f)(B(z))B'(z)^q}\\
&=\overline{\sum_{A\in G} f(AB(z))A'(B(z))^q B'(z)^q}+\sum_{A\in GU} f(AB(z))A'(B(z))^q\overline{B'(z)^q}\\
&=\sum_{A\in GU}\overline{f(A(z))A'(z)^q}+\sum_{A\in G} f(A(z))A'(z)^q\\
&=(\theta_q f)(z).
\end{aligned}$$

By another simple calculation as the above, we know that $\theta_q f$ also satisfies the necessary identity for analytic transformations in E, hence we know that the Pincaré series of f is an element of $A_q^1(D,E)$. To show (3), let T be the characteristic function of a fundamental domain of E. By a simple calculation we have

$$\beta_q f=\theta_q(\beta_q(Tf)) \text{ for any } f \text{ in } L_q^p(D,E).$$

Since Tf is contained in $L_q^p(D)$ and $\beta_q(Tf)\in A_q^p(D)$, we have established (3).

Let D be an invariant union of the components of the region of discontinuity of a (non-elementary) extended group E. Then we have the following theorem.

THEOREM 2. *For* $1\leq p<\infty$ *with* $\frac{1}{p}+\frac{1}{p'}=1$ *the Peterson scalar product*

$$(f,g)_{q,G}=\iint_{X\cup U(X)}\lambda(z)^{2-2q}f(z)\overline{g(z)}dz\wedge d\bar{z},$$

where $f\in A_q^p(D,E)$, $g\in A_q^{p'}(D,E)$ *and* U *is an anti-analytic element of* E, *establishes a real linear topological isomorphism between* $A_q^{p'}(D,E)$ *and the space of imaginary valued bounded linear functionals on* $A_q^p(D,E)$,

PROOF. Let $g\in A_q^{p'}(D,E)$, then

$$(f)=(f,g)_{q,G}, \text{ for all } g\in A_q^p(D,E)$$

is an imaginary valued real linear functional on $A_q^p(D,E)$ with norm $\| l \|\leq \| g \|_{q,p',G}$. Note that $A_q^p(D,E)$ is real linear space but not a complex linear space. Conversely, let l be a bounded real linear functional on $A_q^p(D,E)$ with range in the imaginary numbers. Let T be a characteristic function of a fundamental domain of E. Let

$$TRA_q^p(D,E)=\{\text{real part of } Tf\colon f\in A_q^p(D,E)\}$$

$$TIA_q^p(D,E)=\{\text{imaginary part of } Tf\colon f\in A_q^p(D,E)\}$$

Then the above two spaces are subspaces of $TRA_q^p(D,G)$. For $f\in A_q^p(D,G)$ define the norm

$$\|\mathrm{Real}(Tf)\|=\left(\iint_X \lambda(z)^{2-qp}|f(X)|^p|dz\wedge d\bar{z}|\right)^{1/p} \tag{4}$$

then $TRA_q^p(D,G)$ forms a normed real linear space with norm (4). A holomorphic function is determined by its real part, hence (4) is well defined. If $p\geq 1$ then norm convergence implies uniform convergence of holomorphic functions on compact subsets, hence $TRA_q^p(D,E)$ and $TIA_q^p(D,E)$ are real Banach spaces with the above norm. Define a linear functional l_1 on $TRA_q^p(D,E)$ by l as follows:

$$l_1(Tf_1)=l(f),\ f\in A_q^p(D,E) \text{ and } f_1=\mathrm{Real}(f).$$

Then we know that norm l=norm l_1. We extend l_1 by the Hahn-Banach theorem on $TRA_q^p(D,G)$, and denote the extension also by l_1. We define L on

$$TA_q^p(D,G)=\{Tf\colon f\in A_q^p(D,G)\}$$

by

$$L(Tf)=l_1(Tf_1)+il_1(Tf_2) \text{ for all } f\in A_q^p(D,E)$$

where $f=f_1+if_2$ and $f_1=\mathrm{Real}(f)$. Then

$$\text{norm } L=2 \text{ norm } l_1=2 \text{ norm } l.$$

We observe that $TA_q^p(D,G)$ is a complex linear space with norm (4) and L is a complex linear functional. Now $TA_q^p(D,G)$ is a subspace of $TL_q^p(D,E)$. We extend L on this space and we denote the extended linear functional by the same L. Then by Riesz's Representation Theorem, there is a $g\in TL_q^{p'}(D,E)$ such that

$$L(f)=(f,g)_{q,X} \text{ for all } f\in TA_q^p(D,E).$$

We identify $TL_q^p(D,E)$ and $TA_q^p(D,E)$ with $L_q^p(D,E)$ and $A_q^p(D,E)$ respectively. Suppose $f\in A_q^p(D,E)$ then

$$L(f)=L(f_1+if_2)=l_1(f_1)+il_1(f_2)$$

and

$$(f,g)_{q,X\cup U(X)}=(f,g)_{q,X}-\overline{(f,g)}_{q,X}$$
$$=l_1(f_1)+il_1(f_2)-\{\overline{l_1(f_1+il_1(f_2)}\}$$
$$=2l_1(f_1)=2l(f).$$

We get the above equality since l_1 has its range in the imaginary numbers. Hence we have

$$(f)=(f,\frac{1}{2}g)_{q,X\cup U(X)}$$
$$=(f,\beta_q(\frac{1}{2}g))_{q,X\cup U(X)},$$

where β_q is defined in Lemma 1, since $\beta_q(\frac{1}{2}g)\in A_q^{p'}(D,E)$, it follows that if l is a bounded linear functional on $A_q^p(D,E)$ with range in the imaginary numbers then it corresponds to an element in $A_q^{p'}(D,E)$.

Suppose $g\in A_q^{p'}(D,E)$ and

$$(f)=(f,g)_{q,X\cup U(X)}=0,\ \text{for all } f\in A_q^p(D,E).$$

Then

$$(\beta_q f,g)_{q,X\cup U(X)}=(f,g)_{q,X\cup U(X)}=0$$

for all $f\in L_q^p(D,E)$. Since

$$\overline{(f,g)_{q,X}}=-(f,g)_{q,U(X)}$$

that is $(f,g)_{q,X}$ is real for all $f\in L_q^p(D,E)$, but we know $T(if)\in TL_q^p(D,E)$, hence $(f,g)_{q,X}=0$ for all f and hence $g=0$.

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