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NEW PROOF OF THE HAM-SANDWICH THEOREM

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In this paper we prove the Ham-sandwich theorem by another simple way as taking a lemma. In order to solve it, we caught a critical hint that in W. S. Massey's Algebraic topology "at given time there are two antipodal points on the earth with the same temperature and pressure". We will show more generally this facts by the method of algebraic topology.

MAIN THEOREM. *Given three bounded measurable sets B, H, P in $\boldsymbol{R}^3$, there exists a plane which simultaneously bisects all three (we says B a bread, H a ham and P a pickle, that is, Ham-sandwich).*

LEMMA. *Let t and p be two continuous functions of S^2 to $\boldsymbol{R}$. Then there exist two antipodal points s_0, $-s_0$ in S^2 such that $t(s_0)=t(-s_0)$ and $p(s_0)=p(-s_0)$. (Here we can think of t and p as a function of temperature and of pressure respectively)*

Proof. Define a function $F: S^2 \longrightarrow \boldsymbol{R}^2$ as the follows;

$$F(s)=(t(s)-t(-s), p(s)-p(-s)) \quad \text{for} \quad s \in S^2.$$

Then F is continuous, since t and p are continuous.

Now assume our result is false. Then the vector $F(s)$ is never $(0,0)$ and so may be extended to intersect at a point.

The "prime function" $S^2 \longrightarrow S^1$ is continuous and $(-s)'=-s'$.

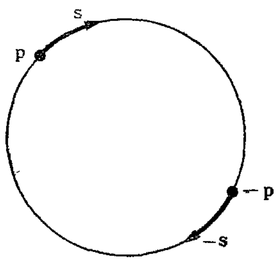
Actually it can be defined as the follows;

$$(s)'=\frac{F(s)}{\|F(s)\|}=s' \quad \text{for} \quad s \in S^2.$$

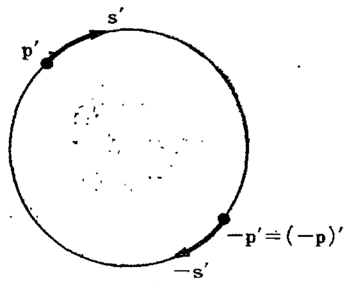
Then it is a antipodal preserving function of S^2 to S^1.

Now as s moves around the equator of S^2 (say "clockwise") from a point p to -p, $-s$ moves antipotally from -p to p and s moves from p′ to (-p′), while $(-s)'$ moves antipotally from (-p)′ to p′.

So the (non-integral) winding number w produced by s' on its trip is equal to the winding number produced by $(-s)'$ on its trip. So if s continued its trip through -p around back to p, the winding number produced by s' on its extended trip would be $2w$, and $2w\neq 0$, since $w\neq 0$.



Equator of S^2



S^1(circle)

So the equator is mapped to a non-zero loop in S^1. We know the prime function which is continuous induces a fundamental group homomorphism (say $'*$) of the fundamental group $\pi_1(S^2)$ to $\pi_1(S^1)$, and S^2 is simply connected.

Therefore the equator loop (say σ) is homotopic to a point, i.e. $[\sigma]$ is a trivial element in $\pi_1(S^2)$. But $[\sigma]'^{*}=[\sigma']$ is a non-trivial element in $\pi_1(S_1)$. It is impossible. (Q. E. D.)

Proof of the main theorem First we will prepare for the proof of our theorem. For $r\in \boldsymbol{R}$, $s\in S^2$, let $\langle rs\rangle$ denote the unique plane which is orthogonal to and intersects the endpoint of the vector $\vec{rs}$.

Call the side of $\langle rs\rangle$ which lies "in the direction of s" the positive side, $\langle rs\rangle^{+}$, and the other side the negative side $\langle rs\rangle^{-}$.

Enumerate the rational points Q^3 of $\boldsymbol{R}^3$ and distribute a small mass among them by assigning a point mass of $\frac{\delta}{2}$ to the first, $\frac{\delta}{4}$ to the second, $\frac{\delta}{8}$ to the third, etc.. We will call the pickle P together with these points masses the finagled pickle F and define a new measure M which will be used for measuring F as follows;

The measure M of F in that region $+\Sigma$ point masses assigned to that region.

Then M has been contrived so that

(1) there is a plane which is orthogonal to a given line and bisecting F,

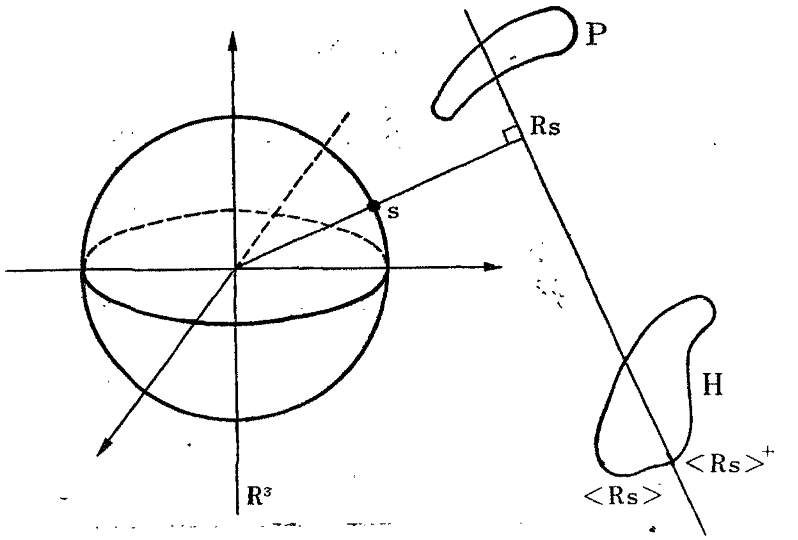
(2) we can associate with each $s \in S^2$ a unique plane $<R(s)s>$, abbreviate $<Rs>$,

(3) $R(-s) = -R(s)$, i. e. antipodal points corresponds to the some plane with opposite orientation,

(4) the association is continuous $(s \longrightarrow R(s))$.

Let $d_1(s)$ be the difference between the amounts of H lying on opposite sides of $<Rs>$, more precisely,

$d_1(s) = m(<Rs>^+ \cap H) - m(<Rs>^- \cap H)$, where m is usual measure. Define d_2 similarly for B.



Thus we have defined two real continuous functions d_1, d_2 on S^2 with the property that $d_i(-s) = -d_i(s)$, $i=1, 2$.

By lemma, there exist antipodal s_0, $-s_0$ in S^2 such that $d_1(s_0) = d_1(-s_0)$, $d_2(s_0) = d_2(-s_0)$, which can only happen if all four of the above quantities are zero. Thus the plane $<R_0 s_0>$ bisects H and B, and almost bisects P (since it bisects F using M).

Furthermore, the error (the difference of the amounts of P on either side of the plane) can be made arbitrarily small by choosing δ sufficient small, i. e. we can find the solution plane by the convergence of the sequence of planes as follows;

apply the above procedure for $\delta=1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\cdots$ to get a sequence of planes $\langle R_i s_i \rangle$ whose errors approache to zero.

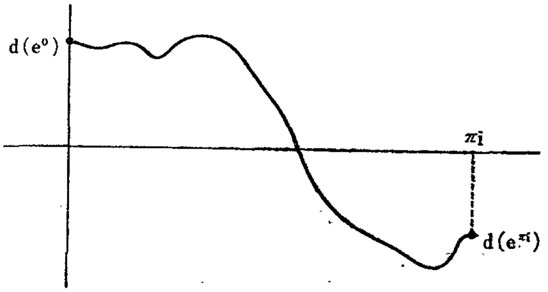
While it is not clear that the sequence of planes converges (i. e. the corresponding vectors $\langle R_i s_i \rangle$ converges), it is clear that the sequence is bounded, since an arbitrarily small fraction of the volume of, say, H lies outside a sphere of sufficient large radius. Thus the sequence has a limit plane and this plane will bisect everything. (Q. E. D.)

Finally 2-dimensional version of the Ham-sandwich theorem can be proven without algebraic topology.

COROLLARY *Given two bounded measurable sets A_1, A_2 in $\boldsymbol{R}^2$, some line simultaneously bisects both.*

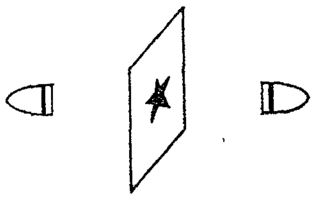
Proof. Proceeding as the begining of the 3-dimensional proof, we can assign in a continuous way to each point s in S^1 a line $\langle Rs \rangle$ which almost bisects A_1 and use this line to construct a real continuous function d on S^1 with the property that $d(-s)=-d(s)$.

Since $d(e^o)=-d(e^{\pi i})$, d must cross the axis in the graph below to yield a line which bisects A_2 and almost bisects A_1.

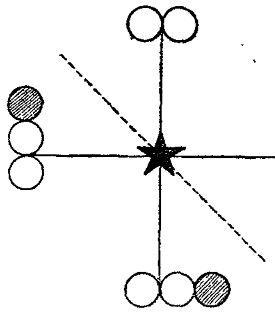


COUNTER EXAMPLES TO TWO FAULTY PROOF

An apparent solution is the plane which goes through the center of mass of each of the 3 components of the sandwich. This solution fails because the matter away from the center is weighted more to take into account its greater leverage. Thus the plane through the center of mass in the picture below does not bisect the pickle.



An analogous solution uses the 3 centers of volume, the corresponding the theory being that any plane through something's center of volume will bisect that thing. This solution fails because the center of volume usually does not exist, as shown below



where the only candiate, lying at the intersection of the solid lines, does not work for the dotted line.

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