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A STUDY ON THE MINIMIZATION OF THE TOTAL ABSOLUTE CURVATURE OF THE IMMERSED SUBMANIFOLDS

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A) For a given immersion $f: M \to E^{n+N}$ of smooth n-manifold M into Euclidean space E^{n+N} of dim. $n+N$, $N \geq 1$, we consider the frame bundle $\pi': F(n, N) \to E^{n+N}$ with the bundle space $F(n, N)$ of all orthonormal frames $xe_1e_2 \cdots e_{n+N}$ at each $x \in E^{n+N}$, and we may consider the adapted frame bundle $\pi: B \to M$ over M. Each fibre over x consists of all the adapted frames $f(x)e_1 \cdots e_n e_{n+1} \cdots e_{n+N}$ where e_i are all tangent $1 \leq i \leq n$, and e_r are normal $n+1 \leq r \leq n+N$ at $f(x)$.

The following is an easy result.

THEOREM 1. *$\pi: B \to M$ is a subbundle of the pull-back of $\pi': F(n, N) \to E^{n+N}$ by the immersion f.*

This theorem may serve for introducing the Cartan's structure equations of $F(n, N)$ into B in the canonical 1-forms ω_A as follows:

(1) $dx = \sum \omega_A e_A$, $de_A = \sum \omega_{AB} e_B$, $\omega_{AB} + \omega_{BA} = 0$

(2) $d\omega_A = \sum \omega_B \wedge \omega_{BA}$, $d\omega_{AB} = \sum \omega_{AC} \wedge \omega_{CB}$

where the indices A, B, and C are range over from 1 to $n+N$. When we employ the restricted indices as

$$1 \leq i, j, k \leq n \quad \text{and} \quad n+1 \leq r, s, t \leq n+N,$$

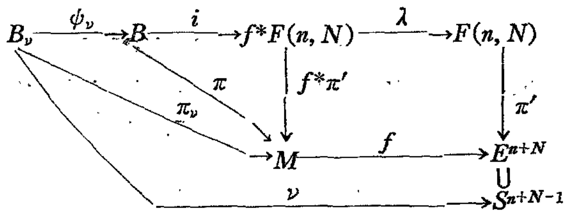
then we have $\omega_r = 0$ and ω_i are independent, and hence we have from (2)

$$\sum \omega_i \wedge \omega_{ir} = 0, \tag{3}$$

and the expression $\omega_{ir} = \sum A_{riB} \omega_B$ of ω_{ir} with the coefficient A_{riB} reduces to

$$\omega_{ir} = \sum A_{rij} \omega_j, \qquad A_{rij} = A_{rji}. \tag{4}$$

Consider the projection $\psi_\nu: B \to B_\nu$ of B onto the unit normal bundle space over M with each of its fibre over x consisting of e_{n+N} at $f(x)$, and we define the map $\nu: B_\nu \to S^{n+N-1}$ of the unit normal bundle space B_ν onto $(n+N-1)$-sphere with the center at the origin of E^{n+N}. We see that the map ν is related with Theorem 1 by the following diagam:



where the bundle $f^*\pi'$ is the pull-back of π' by f, π is the subbundle of $f^*\pi'$ and π_ν is the bundle with the structure group $O(N-1)\cong S^{N-1}$ factored by the projection ϕ_ν.

B) The Lipschitz-Killing curvature $G(x, e_{n+N}) = (-1)^n det(A_{n+Nij})$ on B_ν is defined through the equation

(5) $$\nu^*d\sigma = G(x, e_{n+N})d\mu \wedge d\sigma_{N-1}$$

where $\nu^*d\sigma$ is the differential form on B_ν induced from the volume element $d\sigma$ of S^{n+N-1} by ν, and $d\mu$ and $d\sigma_{N-1}$ are

(6) $$d\mu = \omega_1 \wedge \cdots \wedge \omega_n, \text{ and } d\sigma_{N-1} = \omega_{n+N,\, n+1} \wedge \cdots \wedge \omega_{n+N,\, n+N-1}.$$

The Lipschitz-Killing curvature has the following important properties:

THEOREM 2. *$G(x, e_{n+N})$ is the determinant of the second fundamental form $d\nu(x, e_{n+N}) \cdot df(x)$, and hence a point (x, e_{n+N}) in B_ν is a critical point of ν is equivalent to the fact that $G(x, e_{n+N}) = 0$.*

The proof can be obtained by the derivation of the differentiable map ν defined by $\nu(x, e_{n+N}) = e_{n+N} \in S^{n+N-1}$.

The total absolute curvature $\tau(M, f, E^{n+N})$ of an immersion $f: M \to E^{n+N}$ is defined first in [1] by

(7) $$\tau(M, f, E^{n+N}) = \frac{1}{C_{n+N-1}} \int_{B_\nu} |G(x, e_{n+N})| d\sigma_{N-1} d\mu,$$

if exists, where C_{n+N-1} is the volume of S^{n+N-1}.

C) The minimization of $\tau(M, f, E^{n+N})$ for a fixed oriented compact n-manifold M was originated by the following results of the comparatively small value of $\tau(M, f, E^{n+N})$ in [1].

THEOREM 3. (a) *$\tau(M, f, E^{n+N}) \geq 2$ for any immersion f.*

(b) *$\tau(M, f, E^{n+N}) < 3$ implies that M is homeomorphic to S^n.*

(c) *$\tau(M, f, E^{n+N}) = 2$ if and only if $f(M)$ is a convex hypersurface in some*

E^{n+1}, *a linear subvariety of* E^{n+N}.

The proofs of the theorem depend mainly on the Sard's theorem applied to the differentiable map ν which has the properties of Theorem 2 about the critical points, and on the Reeb's theorem which asserts that if a compact differentiable manifold M has a real-valued differentiable function on it with only two non-degenerate points, then M is homeomorphic to a sphere.

The above theorem reduced to the case of the immersed submanifold in E^3 is found in [4]. The total absolute Gauss-curvature of an abstract compact Riemannian surface M can be defined as

(8) $$\tau(M)=\int_M \frac{|K|}{2\pi}d\mu=\int_{K>0}\frac{K}{2\pi}d\mu-\int_{K<0}\frac{K}{2\pi}d\mu,$$

if we notice that

$$\tau(M)=\tau(M,f,E^3)=\frac{1}{C_2}\int_{B\nu}|K|d\sigma d\mu=\frac{1}{4\pi}\int_M |K|2d\mu.$$

With respect to the Euler-Poincare characteristic $\chi(M)$ of M, the Gauss-Bonnet theorem says:

(9) $$\chi(M)=\frac{1}{2\pi}\int_M Kd\mu$$

From (8) and (9) we obtain

$$-\tau(M)\leq\chi(M)\leq\tau(M),$$

or

(10) $$\tau(M)=\tau(M,f,E^3)\geq|\chi(M)|$$

for an abstract compact surface M, not necessarily orientable, with Riemannian metric.

We notice that (10) is the particular case of the manifold M to be a topological space homeomorphic with a sphere in comparing (a) of Theorem 3.

For an immersion $f: M\rightarrow E^3$ of a compact 2-manifold not necessarily orientable, the following is proved in [4]:

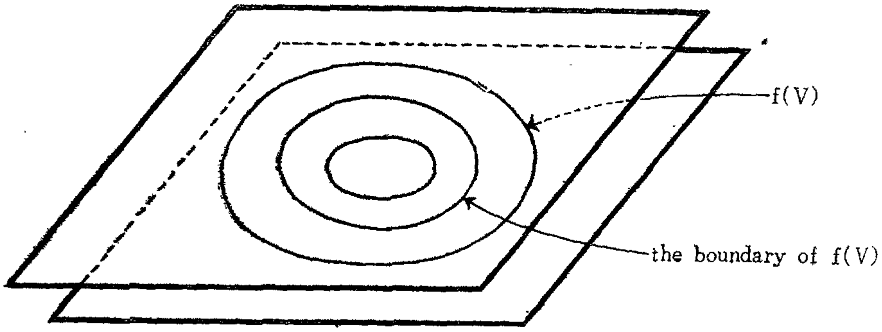
THEOREM 4. $\tau(M,f,E^3)=\int_M \frac{|K|}{2\pi}d\mu\geq 4-\chi(M)$ *for any* f.

An immersion $f: M\rightarrow E^3$ is defined as *convex* immersion if the equality holds in the above theorem. Thus for the case of $\chi(M)=2$, the convex immersion is the case (c) of Theorem 3.

For a torus T ($\chi(T)=0$), the convex immersion $f: T\rightarrow E^3$ with

(11) $$\tau(T,f,E^3)=4-\chi(T)=4$$

necessarily implies that the boundary of $f(V)=f(T)\cap\partial\mathscr{H}(f(T))$, the intersection of the image $f(T)$ and the convex envelope $\partial\mathscr{H}(f(T))$ of $f(T)$ in E^3, consists of two plane convex curves, and the Gauss-curvature K is nonnegative for $x\in V\subset T$ and nonpositive for $T-V$.



By these observation we conclude:

THEOREM 5. *The minimization of* $\tau(M,f,E^3)$ *for a abstract compact manifold* M *is necessarily obtained when the convex envelope* $\partial\mathscr{H}(f(M))$ *of* $f(M)$ *contains all part of* M *for which* $K>0$ *and non of the part for which* $K<0$.

For a given n-manifold M, the total absolute curvature $\tau(M,f,E^{n+N})$ of an immersion f is a function of variables f and N. However for a compact n-manifold M, we have in [2] the following:

(12) $$\tau(M,f,E^{n+N})=\tau(M,f',E^{n+N'}),\ N'>N,$$

where the immersion $f':M\to E^{n+N'}$ *is defined by*

$$f'(x)=(f_1(x),\cdots,f_{n+N}(x),\ 0,0,\cdots,0),\ x\in M$$

when $f:M\to E^{n+N}$ *is defined by*

$$f(x)=(f_1(x),\cdots,f_{n+N}(x)),\ x\in M.$$

Therefore we may consider $\tau(M,f)=\tau(M,f,E^{n+N})$ is a function of one variable only. In order to minimize $\tau(M,f,E^{n+N})$, we are now in the position to investigate the greatest lower bound of $\tau(M,f)$ in variable f.

D) Let z^* be an unit vector in Euclidean vector space E^{n+N}, and z, its dual, the linear function $z:E^{n+N}\to\boldsymbol{R}$ such that $z(y)=y\cdot z^*$ for any $y\in E^{n+N}$. We often identify z and z^*, and write z for z^*. Thus we get the composition $z\circ f:M\to\boldsymbol{R}$ which is a continuous function on M. We write zf for both $z\circ f$ and $z^*\cdot f$.

When an immersion $f:M\to E^{n+N}$ is given, a point z in S^{n+N-1} is called a critical point of f if $\nu : B_\nu \to S^{n+N-1}$ has rank $< n+N-1$ at an inverse image of z. M being compact, the set W of critical points of f is a closed subset of S^{n+N-1}.

Let $\varphi : M \to \boldsymbol{R}$ be a smooth function. A critical point $x \in M$ of φ is called *non-degenerate of index k* if x has a local coordinate functions $\varphi_1, \cdots, \varphi_n : U \to \boldsymbol{R}$ such that in U

$$\varphi = \varphi(x) - \varphi_1^2 - \cdots - \varphi_k^2 + \varphi_{k+1}^2 + \cdots + \varphi_n^2$$

and φ is called *non-degenerate function* if it has only the non-degenerate critical points.

We may ramify the smooth map $f : M \to E^{n+N}$ to the above notion. f shall be called *non-degenerate* if for almost all z in S^{n+N-1} the map zf is non-degenerate (i.e. for all z contained in the subset of the positive Lebesgue measure in S^{n+N-1}).

THEOREM 6. *If $f : M \to E^{n+N-1}$ is an immersion, then it is non-degenerate and moreover for every non-critical point z of f, the map zf has at least two points of M with index 0 at one and with index n at another.*

The proof is due to the Sard's theorem applied to the map $\nu : B_\nu \to S^{n+N-1}$ for the first part and to the fact that every point of S^{n+N-1} is covered at least twice by $\nu : B_\nu \to S^{n+N-1}$ because of the compactness of M.

In the set of smooth-functions $\varphi : M \to \boldsymbol{R}$ on a smooth n-manifold, we introduce the following notations:

$\Phi(M)$ = the set of non-degenerate functions on M.

$\mu_k(M, \varphi)$ = the number of critical points of index k of $\varphi \in \Phi(M)$

$\mu(M, \varphi) = \sum_{k=0}^{n} \mu_k(M, \varphi)$, the number of critical points of $\varphi \in \Phi(M)$

$\gamma_k(M) = \min\{\mu_k(M, \varphi)\ ;\ \varphi \in \Phi(M)\}$

$\gamma(M) = \min\{\mu(M, \varphi)\ ;\ \varphi \in \Phi(M)\}$

Let $\beta_k(M)$ be the maximal rank of the k-th homology group of M for all coefficient rings and the set $\Phi(M)$. Then the Morse inequalities say:

$$\gamma_k(M) \geq \beta_k(M) \tag{13}$$

$$\gamma(M) \geq \sum_{k=0}^{n} \gamma_k(M) \geq \sum_{k=0}^{n} \beta_k(M) = \beta(M) \tag{14}$$

We see through Theorem 6 that to each point z contained in the set G, the complement of the set of the critical points of f in S^{n+N-1}, there assignes a positive integer $\mu(M, zf)$, the number of fibres in B_ν which contains only one preimage of z. If we apply the Sard's theorem to the map ν, the equation (5) and the above notations derive:

$$\tau(M,f)=\tau(M,f,E^{n+N})=\frac{1}{C_{n+N-1}}\int_{B_\nu}|\nu^*d\sigma|=\frac{1}{C_{n+N-1}}\int_{\nu^{-1}(G)}|\nu^*d\sigma|$$

$$=\frac{1}{C_{n+N-1}}\int_{z\in G}\mu(M,zf)\,|d\sigma|=\frac{1}{C_{n+N-1}}\int_{z\in S^{n+N-1}}\mu(M,zf)\,|d\sigma|$$

Thus we have as in [3]

THEOREM 7. *$\tau(M,f)\geq\gamma(M)$, if M is compact.*

This theorem is sharper than the following theorem obtained in [2]:

THEOREM 8. *$\tau(M,f)\geq\beta(M)$, if we refer to the Morse inequality* (14).

E) The greatest lower bound of $\tau(M,f)$ in the variable f is obtained in [3] by

THEOREM 9. *$Inf\{\tau(M,f)$; for any immersion $f\}=\gamma(M)$, for any fixed compact n-manifold M.*

Since we have Theorem 8, the proof is to be accomplished if we can establish a sequence of immersions the total absolute curvatures of which converge to $\gamma(M)$ as argued in [3].

For any immersion $f: M\to E^{n+N}$, and for any positive integers λ, we define $h_\lambda: M\to E^{n+N}\times E^1\equiv E^{n+N+1}$ by $h_\lambda=f\times\lambda\varphi$ where $\varphi\in\Phi(M)$ with $\mu(M,\varphi)=\gamma(M)$. Since $rank(h_\lambda)\geqq rank(f)$, h_λ is an immersion. Thus the established theorem becomes

THEOREM 10. $\lim_{\lambda\to\infty}\tau(M,h_\lambda)=\gamma(M)$.

We shall prove this theorem through the following lemma.

LEMMA 1. *For each integer $\lambda>0$, there exists a closed neighborhood N_λ of the equator L of S^{n+N}, considered $z_0=(0,1)$ as the north pole, such that $z\in N_\lambda$ iff $\mu(M,zh_\lambda)\neq\gamma(M)$.*

Proof. Consider the immersion $h_\lambda=f\times\lambda\varphi: M\to E^{n+N+1}$. If we choose $z_0=(0,\cdots,0,1)\equiv(0,1)\in S^{n+N}\subset E^{n+N}\times E^1$, $z_0h_\lambda=\lambda\varphi$, and hence z belongs to the open subset G of S^{n+N} and z_0h_λ belongs to $\Phi(M)$, having the number of non-degenerate critical points eaqual to $\gamma(M)$ for each $\lambda>0$. Since zh_λ is continuous linear in the variable z of E^{n+N+1}, so are $d(zh_\lambda)$ and the Hessian of zh_λ, particularly at z_0. Choosing $\lambda=1$, there exists $\varepsilon>0$ such that

$$D=\{z=(\omega,\xi)\,|\,\omega\in E^{n+N},\ \xi\in E^1\}$$

is an open neighborhood of $z_0=(0,1)$ in $E^{n+N}\times E^1$ such that $|z-z_0|<\sqrt{\varepsilon}$ implies $\mu(M,zh_1)=\gamma(M)$. If we denote

$$D_1=\{z\,|\,z=(\omega,\xi)\in D,\ \xi=1\}$$

then for any $z\in D_1$, z satisfies that $|z-z_0|<\sqrt{\varepsilon}$ or equivalently $\omega^2<\varepsilon$. That is, $zh_1=\omega f+\varphi$ belongs to $\Phi(M)$ with $\mu(M,zh_1)=\gamma(M)$ if $\omega^2<\varepsilon$.

If we put

$\psi=\lambda zh_1=\lambda\omega f+\lambda\varphi$ for any positive integer λ, then $\psi\in\Phi(M)$ with $\mu(M,\psi)=\gamma(M)$ if $\lambda^2\omega^2<\lambda^2\varepsilon$.

On the other hand, for any $z\in D\cap S^{n+N}$,

$$zh_\lambda=\omega f+\lambda\xi\varphi.$$

Since ξ can be assumed non-zero, we have equivalently

$$\frac{1}{\xi}zh_\lambda=\frac{\omega}{\xi}f+\lambda\varphi.$$

Hence zh_λ belongs to $\Phi(M)$ with $\mu(M,zh_\lambda)=\gamma(M)$ if $\left(\frac{\omega}{\xi}\right)^2<\lambda^2\varepsilon$ or equivalently $1/\sqrt{1+\varepsilon\lambda^2}<\xi^2$.

We denote D_λ' the set of points (ω,ξ) in $E^{n+N}\times E^1$ such that

$$-1/\sqrt{1+\varepsilon\lambda^2}\leq\xi\leq1/\sqrt{1+\varepsilon\lambda^2}$$

If $z\in D_\lambda'\cap S^{n+N}$, then $\gamma(M)\neq\mu(M,zh_\lambda)<\infty$, since $z\in D_\lambda'\cap S^{n+N}$ if and only if $\left|\frac{\omega}{\xi}\right|^2\geq\varepsilon\lambda^2$.

We have now only to put $N_\lambda=D_\lambda'\cap S^{n+N}$ to get the required neighborhood.

Proof of Theorem 10. For an arbitrary positive integer λ, it is clear that $N_{\lambda+1}\subset N_\lambda$ by the construction. Hence $\{N_\lambda : \lambda\in Z^+\}$ is a nested sequence of bounded closed subsets each containg the equator L of S^{n+N}. It is clear that it converges to the equator L of S^{n+N}, i.e. $\lim_{\lambda\to\infty}N_\lambda=L$.

Hence

$$\lim_{\lambda\to\infty}\tau(M,h_\lambda)=\lim_{\lambda\to\infty}\frac{1}{C_{n+N}}\int_{S^{n+N}}\mu(M,zh_\lambda)\,|d\sigma_{n+N}|$$

$$=\lim_{\lambda\to\infty}\left\{\frac{1}{C_{n+N}}\int_{S^{n+N}-N_\lambda}\gamma(M)\,|d\sigma_{n+N}|+\frac{1}{C_{n+N}}\int_{N_\lambda}\mu(M,zh_\lambda)\,|d\sigma_{n+N}|\right\}$$

$$=\frac{1}{C_{n+N}}\int_{S^{n+N}-L}\gamma(M)\,|d\sigma_{n+N}|+0$$

$$=\frac{1}{C_{n+N}}\int_{S^{n+N}}\gamma(M)\,|d\sigma_{n+N}|=\gamma(M).$$

References

[1] S. S. Chern and R. K. Lashof, *On the total curvature of immersed manifolds*, Amer. J. of Math. 79(1957), 306-313.

[2] ______ and ______ , *On the total curvature of immersed manifolds* II, Michigan Math. Journ. 5(1958), 5-12.

[3] N. H. Kuiper, *Immersions with minimal total absolute curvature*, Collegue de géometrie differentialle globale, Brexelles (1958), 75-88.

[4] ______ , *On surfaces in Euclidean 3-spaees*, Bullitin de la societe Math. Belgique(1960), 5-22.

[5] ______ , *La courbre d'indice k et les applications convexes Séminaire de topologie et gêometrie differentiallê*, C. Ehresmann (1959), 1-5.

[6] ______ , *Minimal total absolute curvature for immersions*, Invent Math. **10** (1970), 209-238.

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