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# ON COMPLETE VECTOR LATTICES OF ORDER BOUNDED LINEAR MAPPINGS

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## Introduction.

When $E$ is a vector lattice and $F$ is an order complete vector lattice, the set of all order bounded linear mappings from $E$ into $F$, which is denoted by $\mathcal{L}(E, F)$, forms an order complete vector lattice. (cf. [1]) We tried to find out some properties of $\mathcal{L}(E, F)$ and its subspaces.

## 1. Preliminaries.

When a real vector space $E$ is equipped with a partial order $\leq$ having the following properties: if $x\leq y$, then $x+z\leq y+z$ for any $x, y$ and $z$ of $E$ and if $\alpha$ is a positive real number, then $x\leq y$ implies $\alpha x\leq \alpha y$, we call $E$ an *ordered vector space*. We shall denote by $E^+$, the set $\{x\in E | x\geq\theta\}$. An ordered vector space $E$ is called a *vector lattice* if there exist the supremum $x\vee y$ and the infimum $x\wedge y$ for every pair of elements of $E$. Suppose that $E$ is a vector lattice and $x\in E$. We define $x^+=x\vee\theta$, $x^-=(-x)^+$ and call them the ***positive prat*** and the ***negative part***, respectively, of $x$. The ***absolute value*** $|x|$ of $x$ is defined as $x\vee(-x)$. It is easy to see that $x=x^+-x^-$; hence $E=E^+-E^-$ and that $|x|=x^++x^-$. Let $E$ be a vector lattice. A subset $S$ of $E$ is called *order bounded* if it is contained in some order interval $[x, y]=\{z\in E | x\leq z\leq y\}$. $E$ is said to be *order complete* if every order bounded subset of $E$ has the supremum and the infimum in $E$. A net $\{x_\alpha\}$ is said to *order converge* to $x$ in a vector lattice $E$ if there exist an increasing net $\{y_\alpha\}$ and a decreasing net $\{z_\alpha\}$ such that $\sup\{y_\alpha\}=x=\inf\{z_\alpha\}$ and for any $x_\alpha$, there exist $y_\beta$ and $z_\gamma$ such that $y_\beta\leq x_\alpha\leq z_\gamma$. A subset $S$ of $E$ is said to be *closed* if $S$ contains the limit of every order convergent net in $S$. Suppose that $E$ is an ordered vector space and that $E$ is the direct sum of two linear subspaces $M$ and $N$. $E$ is called the *order direct sum* of $M$ and $N$ if $x\geq\theta$ and $x=x_1+x_2 (x_1\in M,\ x_2\in N)$ imply $x_1\geq\theta$ and $x_2\geq\theta$. A linear subspace $I$ of a vector lattice $E$ is called a *lattice ideal* if $y \in I$ whenever $x \in I$ and $|y| \leq |x|$. A lattice ideal $I$ of $E$ is a *band* if $I$ contains the supremum of every subset of $I$ that is bounded above in $E$. Let $E$ and $F$ be vector lattices. A linear mapping $\varphi : E \longrightarrow F$ is *order bounded* (order continuous, respectively) if it maps every order bounded set (order convergent net) in $E$ to an order bounded set (order convergent net) in $F$. The set of all order bounded (order continuous, respectively) linear mappings from $E$ into $F$ is denoted by $\mathcal{L}(E, F)$ ($\mathcal{L}_n(E, F)$, respectively) will be denoted by $E^b$ ($E^c$, respectively). A linear mapping $\varphi : E \longrightarrow F$ is *positive* if the image of every positive vector under $\varphi$ is positive. This induces a partial order $\geq$ in $\mathcal{L}(E, F)$, which is defined by $\varphi \geq \psi$ if $\varphi - \psi$ is positive. F. Riesz verified the following facts: (cf. [1]) If $E$ is a vector lattice and $F$ is an order complete vector lattice, then $\mathcal{L}(E, F)$ forms an order complete vector lattice and $\mathcal{L}_n(E, F)$ is a closed ideal of $\mathcal{L}(E, F)$. Every closed ideal of an order complete vector lattece is a band. $I$ is a band in an order complete vector lattice $E$ if and only if $E$ is the order direct sum of $I$ and another band $I'$.

## 2. Extension of order bounded linear mappings.

When $E$ and $F$ are complete vector lattices and $I$ is a closed ideal in $E$, every order bounded linear mapping: $I \longrightarrow F$ can be extended to the whole space $E$ preserving the order boundedness.

LEMMA. *The projection of every order bounded set of $E$ into $I$ is order bounded in $I$ and the projection of every order convergent sequence into $I$ is order convergent in $I$.*

*Proof*. Every assertion follows from the fact that a closed ideal is a band and hence $E$ is represented as an order direct sum $E = I \oplus I'$.

THEOREM 1. *The natural restriction map $\theta : \mathcal{L}(E, F) \longrightarrow \mathcal{L}(I, F)$ defined by $\theta(\varphi) = \varphi | I$ $(\varphi \in \mathcal{L}(E, F))$ is a surjective linear mapping and $\theta(\mathcal{L}(E, F)^+) = \mathcal{L}(I, F)^+$.*

*Proof*. For the first assertion of the theorem, it suffices to show that any $\varphi \in \mathcal{L}(I, F)^+$ can be extended to an element of $\mathcal{L}(E, F)^+$, due to the fact that both $\mathcal{L}(E, F)$ and $\mathcal{L}(I, F)$ are complete vector lattices and hence they are decomposed into positive and negative parts. Since $I$ is a band, $E$ is represented as an order direct sum $E = I \oplus I'$. Define $\bar{\varphi} : E \longrightarrow F$ by $\bar{\varphi}(x) = \varphi(x_1)$, where $x$

$=x_1+x_2$, ($x_1 \in I$ and $x_2 \in I'$). Clearly $\phi$ maps every positive vector to a positive vector; hence $\phi \in \mathcal{L}(E, F)^+$. This gives the required extension. $\theta(\mathcal{L}(E, F)^+) = \mathcal{L}(I, F)^+$ is obvious since $I^+ \subseteq E^+$.

### 3. Dual mappings.

For any $\varphi \in \mathcal{L}(E, F)$, define $\varphi^t : F^b \longrightarrow E^b$ by $\varphi^t(f) = f \circ \varphi$ ($^\forall f \in F^b$). Clearly $\varphi^t(f) \in E^b$, and hence $\varphi^t$ is well defined. Furthermore, $\varphi^t(f) \in E^c$ if $f \in F^c$ and $\varphi \subset \mathcal{L}_n(E, F)$.

LEMMA. *If $\varphi \in \mathcal{L}(E, F)$, then $\varphi^t$ is $\sigma(F^b, F) - \sigma(E^b, E)$ continuous.* (cf. [2])

THEOREM 2. *If $[a, b]$ is an order interval in $F^b$, then $\varphi^t\ [a, b]$ is weak*-closed in $E^b$.*

*Proof.* We shall first show that $[a, b]$ is weak*-compact in $F^b$. We note that $\sigma(F^b, F)$ is the topology on $F^b$ induced by the product topology on $R^F = \prod_{x \in F} R_x$, where each $R_x$ is the real line. Therefore, it suffices to show that $[a, b]$ is compact in $R^F$. Let a net $\{\alpha_i\}$ converges to $\alpha$ in $R^F$, where each $\alpha_i \in [a, b]$; or equivalently, $\alpha_i(x) \longrightarrow \alpha(x)$ for all $x \in F$. Then clearly $\alpha$ is linear. Moreover, for any $x \in F^+$ we have $a(x) \leq \alpha_i(x) \leq b(x)$, which implies $a(x) \leq \alpha(x) \leq b(x)$. Hence $\alpha$ is an order bounded linear functional and $\alpha \in [a, b]$. Therefore, $[a, b]$ is closed in $R^F$. We shall reach the conclusion by showing that $[a, b]$ is a subset of a compact subset of $R^F$. Denote the set $\{f \in R^F \mid {}^\forall x \varepsilon F,\ {}^\exists \alpha \in [a, b] : \alpha(x) = f(x)\}$ by $[\![a, b]\!]$. Since $[a, b](x) = [a, b](x^+ - x^-)$ is a subset of $[a, b](x^+) - [a, b](x^-)$ which is bounded in $R^I$, $\overline{[a, b](x)}$ is compact. Then $\prod_{x \in F} \overline{[\![a, b]\!](x)} = \prod_{x \in F} \overline{[a, b]\ (x)}$ is compact by Tychonoff and it contains $[a, b]$. Hence $[a, b]$ is compact in $R^F$. By lemma $\varphi^t[a, b]$ is weak*-compact in $E^b$, and hence weak*-closed since $\sigma(E^b, E)$ is Hausdorff. This completes the proof.

COROLLARY. *If $[a, b]$ is an order interval in $F^c$, then $\varphi^t[a, b]$ is weak*-compact in $E^c$.*

*Proof.* An order interval of $F^c$ is identical with that of $F^b$. For, consider the order interval $[a, b]$ in $F^b$, where $a, b \in F^c$. If $z \in [a, b]$, then $\theta \leq b - z \leq (b - z) + (z - a) = b - a \in F^c$. Since $F^c$ is a band in $F^b$, $b - z \in F^c$ and hence $z \in F^c$. Therefore, $[a, b] \subset F^c$. By theorem 2 $\varphi^t[a, b]$ is weak*-compact in $E^b$.

But since $\varphi^t[a,b]\subset E^c$ and $\sigma(E^c, E)$ is the subspace topology induced by $\sigma(E^b, E)$, $\varphi^t[a,b]$ is weak*-compact in $E^c$, and hence weak*-closed. It is not difficult to show that all the theory discussed in this work can be applied also in the $\mathcal{L}_n(E,F)$.

## References

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