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SUBPARACOMPACT SPACES AND PSEUDO-CLOSED MAPS

BY MOO HA WOO

It is shown that a locally semi-stratifiable subparacompact space is semi-stratifiable. If a space X is the union of the locally finite family of closed subparacompact subspaces, then X is also subparacompact. The image of a countable subparacompact (metacompact) space under a pseudo-closed mapping is countable subparacompact (metacompact).

Notation and terminology will follow that of J. L. Kelley [8] and all mappings will be continuous and surjective. We denote the complement of a subset A of a topological space by $\mathscr{C}A$.

1. Properties of subparacompact spaces.

DEFINITION 1. 1. [3] A topological space is said to be *subparacompact* if every open cover has a σ-discrete closed refinement.

This class of spaces was introduced by McAuley [10] and more recently has been studied by Burke [3]. Burke showed that a space is subparacompact if and only if every open cover of the space has a σ-locally finite closed refinement.

DEFINITION 1. 2. [5] A topological space X is a *semi-stratifiable* space if, to each open set $U\subset X$, one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of closed subsets of X such that

(a) $\bigcup_{n=1}^{\infty} U_n=U$,

(b) $U_n\subset V_n$ whenever $U\subset V$.

(If, in addition, the space X satisfies (c) $\bigcup_{n=1}^{\infty}\text{Int } U_n=U$, then X is *stratifiable* [2].)

A correspondence $U\longrightarrow\{U_n\}_{n=1}^{\infty}$ is a *semi-strafication* (*stratification* for the space X whenever it satisfies the conditions (a), (b) (and (c)). Ceder [4] introduced M_3-spaces and Borges [2] renamed them "stratifiable", while Creede

[5] (Kofner [9])studied semi-stratifiable spaces (pseudo-stratifiable spaces [that is, a spaces has a σ-cushioned pair-network]). We know that those two spaces are equivalent (see [2] p. 2)

We know that every paracompact space is subparacompact and every semi-stratifiable space is subparacompact [5]. Ceder has shown that any locally stratifiable paracompact space is stratifiable. We can obtain analogous result as follow:

THEOREM 1.3. *A locally semi-stratifiable subparacompact space is semi-stratifiable.*

Proof. Let X be a locally semi-stratifiable subparacompact space. Then for each $x \in X$, there exists on open neighborhood U_x such that U_x is semi-stratifiable. Since X is subparacompact, the open cover $\{U_x | x \in X\}$ has a σ-discrete closed refinement $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$. For each $F_\alpha \in \mathcal{A}$, F_α is contained in U_x for some $x \in X$. Since semi-stratifiable spaces is hereditary, F_α is semi-stratifiable.

Let G be an open set in X. Then, since $G \cap F_\alpha$ is open in $F_\alpha \in \mathcal{A}_n$ for each $n \in N$ and each α, there is a semi-stratification $G \cap F_\alpha \longrightarrow \{[G \cap F_\alpha]_m\}_{m=1}^{\infty}$. Let $G_{n,m} = \bigcup \{[G \cap F_\alpha]_m | F_\alpha \in \mathcal{A}_n\}$. Then $G_{n,m}$ is closed in X, since $[G \cap F_\alpha]_m$ is closed in X and $\mathcal{A}_n$ is discrete. It is easy to check that $G = \bigcup_{n,m=1}^{\infty} G_{n,m}$ and $G_{n,m} \subset G^*_{n,m}$ whenever $G \subset G^*$. Thus the correspondence $G \longrightarrow \{G_{n,m}\}_{n,m=1}^{\infty}$ is a semi-stratification for the space. X.

It is well known that a locally first countable space is first countable. With the aid of corollary 1.4 of [5], we have the following corollary.

COROLLARY 1.4. *A locally semi-metrizable subparacompact space is semi-metrizable.*

THEOREM 1.5. *If a space X is the union of a locally finite family of closed subparacompact subspaces, then X is also subparacompact.*

Proof. Let X be the union of a locally finite family $\{F_\alpha\}$ of closed subparacompact spaces. Let $\{U_\beta\}$ be an open cover of X. Then $\{F_\alpha \cap U_\beta\}$ is an open cover of F_α. Since F_α is subparacompact, there exists a σ-discrete closed refinement $\mathcal{A}^\alpha = \bigcup_{n=1}^{\infty} \mathcal{A}_n^\alpha$ of $\{F_\alpha \cap U_\beta\}$. Let $\mathcal{A}_n = \bigcup_\alpha \mathcal{A}_n^\alpha$. Then $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$ is a σ-locally finite closed refinement of $\{U_\beta\}$.

It is sufficient to show that $\mathcal{A}_n$ is a locally finite family. For each $x \in X$, there is an open U_x containing x which intersects only finitely many members of $\{F_\alpha\}$ say F_{α_1}, $F_{\alpha_2}, \cdots, F_{\alpha_n}$. Then we may assume that x is contained in each F_{α_i}, $i=1,2,\cdots,n$. Since $\mathcal{A}_n^{\alpha_i}$ is discrete and $x \in F_{\alpha_i}$, $i=1,2,\cdots,n$, there is an open neighborhood V_{α_i} of x in X such that $V_{\alpha_i} \cap F_{\alpha_i}$ intersects at most one element of $\mathcal{A}_n^{\alpha_i}$ for $i=1,2,\cdots,n$. Let $G_x = U_x \cap (\bigcap_{i=1}^{n} V_{\alpha_i})$. Then G_x is a neighborhood of x intersecting at most finite elements of $\mathcal{A}_n$. Thus X is subparacompact.

COROLLARY 1.6. *If X is the union of a locally finite family of closed semi-stratifiable subspaces, then X is semi-stratifiable.*

Proof. It can be proved by Theorem 1.3, Theorem 1.5 and Theorem 2.8 of [4].

DEFINITION 1.7. [11]. Let ω be an open cover of a space X. A mapping f from the space X onto some space Y is called an ω-mapping if for each point y in Y, there exists a neighborhood 0_y such that $f^{-1}(0_y)$ is contained in an element of the cover ω.

Pareek [11] proved that if every open cover ω of the space X there exists an ω-mapping $f: X \longrightarrow Y$ onto a s-paracompact space Y, then X is a s-paracompact space. Now we can obtain analogous results as follow:

THEOREM 1.8. *If for every open cover ω of the space X there exists an ω-mapping $f: X \longrightarrow Y$ onto a subparacompact space Y, then X is a subparacompact space.*

Proof. Let ω be an open cover of the space X. Then by the hypothesis there exists an ω-mapping $f: X \longrightarrow Y$ onto a subparacompact space Y. For each y in Y, let O_y be an open neighborhood of y such that $f^{-1}(0_y)$ is contained in some $U \in \omega$. Evidently, $\{0_y \mid y \in Y\}$ is an open cover of Y. Since Y is subparacompact, there exists a σ-discrete closed refinement $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$. Let $\mathcal{A}_n^* = \{f^{-1}(B) \mid B \in \mathcal{A}_n\}$ for each $n \in N$. Since f is continuous and $\mathcal{A}_n$ is discrete, $\mathcal{A}_n^*$ is a discrete closed collection for each $n \in N$. For each $f^{-1}(B)\mathcal{A}_n^*$, $f^{-1}(B) \subset f^{-1}(0_y)$ for some $y \in Y$ and $f^{-1}(0_y) \subset U$ for some $U \in \omega$. It is trivial that $\mathcal{A}^* = \bigcup_{n=1}^{\infty} \mathcal{A}_n^*$ is a cover of X. Thus X is subparacompact.

COROLLARY 1.9. *If $f: X \longrightarrow Y$ is a closed mapping from a space X onto a subparacompact space Y such that for each open cover ω and each y in Y, $f^{-1}(y)$ is contained in an elementof ω, then X is subparacompact.*

2. Pseudo-closed mappings.

In this section, a class of mappings, called pseudo-closed mappings, is introduced. Many topological properties be preserving under closed mappings are also preserved under pseudo-closed mappings.

DEFINITION 2.1. A mapping $f: X \longrightarrow Y$ is pseudo-closed if for each open $U \subset Y$ and any closed $F \subset f^{-1}(U)$, it follows that $cl(f(F)) \subset U$.

LEMMA 2.2. *A mapping $f: X \longrightarrow Y$ is closed if and only if for each $A \subset Y$ and any closed $F \subset f^{-1}(A)$, it follows that $cl(f(F)) \subset A$,*

Proof. If: Let F be a closed set in X, then we let $A = f(F)$. Since F is contained in $f^{-1}(A)$, by hypothesis we have that $cl(f(F)) \subset A$. Therefore f is closed.

Only if: It is trivial.

REMARK. It is trivial that every closed mapping is pseudo-closed but the converse is not true.

THEOREM 2.3. *If $f: X \longrightarrow Y$ is a mapping and Y is a T_1-space, then f is closed if and only if it is pseudo-closed.*

Proof. Let $A(=\mathcal{C}B)$ be a subset of Y and any closed $F \subset f^{-1}(A)$. Since $f^{-1}(A) = f^{-1}(\mathcal{C}B) = \bigcap_{y \in B} f^{-1}(\mathcal{C}y)$, we have that $F \subset f^{-1}(\mathcal{C}y)$ for each $y \in B$. Since f is pseudo-closed, $cl(f(F)) \subset \mathcal{C}y$ for each $y \in B$. Therefore $cl(f(F)) \subset \mathcal{C}B = A$. Thus f is closed by Lemma 3.2.

THEOREM 2.4. *If $f: X \longrightarrow Y$ is pseudo-closed and X is semi-stratifiable, then Y is semi-stratifiable.*

Proof. Let U be an open subset of Y. Then, since f is continuous, $f^{-1}(U)$ is an open set in X. Since X is semi-stratifiable, there is a semi-stratification $f^{-1}(U) \longrightarrow \{[f^{-1}(U)]_n\}_{n=1}^{\infty}$. Since f is pseudo-closed and $[f^{-1}(U)]_n$ is closed in

$f^{-1}(U)$, we have $cl(f([f^{-1}(U)]_n)\subset U$. We also have that $U=\bigcup_{n=1}^{\infty} cl(f([f^{-1}(U)]_n)$, since $f^{-1}(U)=\bigcup_{n=1}^{\infty}[f^{-1}(U)]_n$. It is trivial that $cl(f([f^{-1}(U)]_n))\subset cl(f([f^{-1}(V)]_n))$ whenever $U\subset V$. The correspondence $U\longrightarrow\{cl(f([f^{-1}(U)]_n))\}_{n=1}^{\infty}$ is a semi-stratification for Y.

COROLLARY 2.5. (Creede) *If $f: X\longrightarrow Y$ is closed and X is semi-stratifiable, then Y is semi-stratifiable.*

Hodel [6] showed that a topological space X is countably subparacompact if and only if it satisfies this condition: given a countable open cover $\{U_n\}$ of X, there is a countable closed cover $\{F_{nj}\}$ of X with $F_{nj}\subset U_n$ for all n and j.

THEOREM 2.6. *If $f: X\longrightarrow Y$ is pseudo-closed and X is countably subparacompact, then Y is countably subparacompact.*

Proof. Let $\{U_n\}$ be a countable open cover of Y. Then $\{f^{-1}(U_n)\}$ is a countable open cover of X. Since X is countably subparacompact, there exists a countable closed cover $\{F_{nj}\}$ with $F_{nj}\subset f^{-1}(U_n)$. Since f is pseudo-closed and $F_{nj}\subset f^{-1}(U_n)$, we have that $cl(f(F_{nj}))\subset U_n$ for each n and j. Thus we have a countable closed cover $\{cl(f(F_{nj}))\}$, of Y with $cl(f(F_{nj}))\subset U_n$. Hence Y is countably subparacompact.

Ishikawa [7] showed that a topological space is countably metacompact if and only if for a given countable decreasing chain $\{F_n\}$ of nonempty closed sets with vacuous intersection, there exists a countable decreasing chain $\{G_n\}$ of open sets with a vacuous intersection such that $F_n\subset G_n$.

THEOREM 2.7. *If $f: X\longrightarrow Y$ is pseudo-closed and X is countably metacompact, then Y is countably metacompact.*

Proof. Let $\{F_n\}$ be a countable decreasing chain of closed sets of Y with void intersection. Then, since f is continuous, $\{f^{-1}(F_n)\}$ is a countable decreasing chain of closed sets of X with empty intersection. Since X is countably metacompact, there is a decreasing chain $\{U_n\}$ of open sets such that $f^{-1}(F_n)\subset U_n$ for each n. Now, let $G_n=\mathscr{C}[cl(f(\mathscr{C}U_n))]$ for each n. Then we have that $G_n\supset F_n$ and $G_n\supset G_{n+1}$. Since f is pseudo-closed, $\mathscr{C}F_n$ is open in Y and $\mathscr{C}U_n\subset\mathscr{C}[f^{-1}(F_n)]=f^{-1}(\mathscr{C}F_n)$, we obtain $cl(f(\mathscr{C}U_n))\subset\mathscr{C}F_n$. Thus we have $F_n\subset\mathscr{C}[cl(f$

$(\mathscr{C}U_n))]=G_n$. It is obvious that $\bigcap_{n=1}^{\infty}G_n=\phi$, since $f^{-1}(\bigcap_{n=1}^{\infty}G_n)=\bigcap_{n=1}^{\infty}f^{-1}(G_n)\subset\bigcap_{n=1}^{\infty}U_n$.

COROLLARY 2.8. (Banerjee). *If $f: X\longrightarrow Y$ is closed and X is countably metacompact, then Y is countably metacompact.*

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Soong Jun University