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ON HYPERSURFACES OF A K-CONTACT METRIC MANIFOLD AS FIBRED SPACES

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0. Introduction.

A structure called an (f, g, u, v, λ)-structure induced on hypersurfaces of an almost contact metric manifold has been studied by many authors (cf. [1], [5], [6]).

K. Yano and M. Ako ([1]) studied hypersurfaces of an odd-dimensional sphere as fibred spaces with one or two-dimensional fibres.

Main purpose of the present paper is to extend the theory developed in [1]. In detail we investigate hypersurfaces with a complemented f-structure immersed in a K-contact metric manifold as fibred spaces with one-dimensional or two-dimensional fibres.

1. Preliminaries.

Let $\tilde{M}^{2n+1}$ be a $(2n+1)$-dimensional almost contact metric manifold covered by a system of coordinate neighborhoods $\{\tilde{U} ; y^\kappa\}$, where and here the indices $\kappa, \lambda, \mu, \nu, \cdots$ run over the range $\{1, 2, \cdots, 2n+1\}$ and let $(F_\lambda{}^\kappa, G_{\mu\lambda}, v^\lambda)$ be an almost contact metric structure, that is,

$$(1.1)\qquad \begin{aligned} &F_\mu{}^\kappa F_\lambda{}^\mu = -\delta_\lambda^\kappa + v_\lambda v^\kappa, \qquad G_{\gamma\beta} F_\mu{}^\gamma F_\lambda{}^\beta = G_{\mu\lambda} - v_\mu v_\lambda, \\ &v_\kappa F_\lambda{}^\kappa = F_\lambda{}^\kappa v^\lambda = 0, \qquad v_\lambda v^\lambda = 1. \end{aligned}$$

A K-contact metric manifold is characterized by an almost contact metric manifold with a Killing vector field v^λ such that

$$(1.2)\qquad \nabla_\mu v^\lambda = F_\mu{}^\lambda.$$

It is well-known that the following identities are valid in a K-contact metric manifold (cf. [7]), i.e.,

$$(1.3)\qquad \nabla_\lambda F_{\mu\nu} + R_{\kappa\lambda\mu\nu} v^\kappa = 0,$$

$$(1.4)\qquad R_{\kappa\lambda\mu\nu} v^\kappa v^\nu = G_{\lambda\mu} - v_\lambda v_\mu,$$

where $R_{\kappa\lambda\mu\nu}$ are covariant components of Riemannian curvature tensor of $\tilde{M}^{2n+1}$.

Let M^{2n} be a $2n$-dimensional differentiable manifold which is covered by a system of coordinate neighborhoods $\{U ; x^h\}$ $(h, i, j, k, \cdots = 1, 2, \cdots, 2n)$, and which is immersed in $\tilde{M}^{2n+1}$ as a hypersurface by the equations

$$(1.5)\qquad y^\kappa = y^\kappa(x^h).$$

We put $B_i{}^\kappa = \partial_i y^\kappa$ and choose a unit vector C^κ of $\tilde{M}^{2n+1}$ normal to M^{2n} in such a way that $2n+1$ vectors $B_i{}^\kappa$ and C^κ give the positive orientation of $\tilde{M}^{2n+1}$.

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The transforms $F_\lambda{}^\kappa B_i{}^\lambda$ of $B_i{}^\lambda$ by $F_\lambda{}^\kappa$ can be expressed as

$$F_\lambda{}^\kappa B_i{}^\lambda = f_i{}^h B_h{}^\kappa + u_i C^\kappa, \tag{1.6}$$

where $f_i{}^h$ is a tensor field of type (1, 1) and u_i is a 1-form of M^{2n}. Similarly, the transform $F_\lambda{}^\kappa C^\lambda$ of C^λ by $F_\lambda{}^\kappa$ can be written as

$$F_\lambda{}^\kappa C^\lambda = -u^i B_i{}^\kappa, \tag{1.7}$$

where $u^i = u_j g^{ji}$, $(g^{ji}) = (g_{ji})^{-1}$, g_{ji} being the Riemannian metric on M^{2n} induced from that of $\tilde{M}^{2n+1}$. In the sequel we assume that the vector field v^κ is tangent to M^{2n}, that is,

$$v^\kappa = B_i{}^\kappa v^i, \tag{1.8}$$

where v^i is a vector field of M^{2n}.

Applying $F_\kappa{}^\mu$ to (1.6), (1.7) and (1.8) respectively, we obtain

$$\begin{gathered} f_i{}^t f_t{}^h = -\delta_i^h + u_i u^h + v_i v^h, \\ g_{ts} f_j{}^t f_i{}^s = g_{ji} - u_j u_i - v_j v_i, \\ u_t f_j{}^t = v_t f_j{}^t = 0, \qquad u_i u^i = v_i v^i = 1, \qquad u_i v^i = 0. \end{gathered} \tag{1.9}$$

Equations (1.9) show that M^{2n} admits a complemented f-structure (cf. [5]). We now put $f_j{}^t g_{ti} = f_{ji}$, then

$$f_{ji} = -f_{ij} \tag{1.10}$$

by virtue of the second equation of (1.9).

For the hypersurface M^{2n}, the equations of Gauss and those of Weingarten are respectively

$$\nabla_j B_i{}^\kappa = k_{ji} C^\kappa, \qquad \nabla_j C^\kappa = -k_j{}^i B_i{}^\kappa, \tag{1.11}$$

where k_{ji} are second fundamental tensor field of M^{2n} and $k_j{}^i = k_{jt} g^{ti}$ and denote ∇_j by the operator of covariant differentiation with respect to the Christoffel symbols $\{{}_j{}^h{}_i\}$ formed with g_{ji}.

Differentiating (1.8) covariantly along M^{2n} and taking account of (1.2) and (1.11), we have

$$F_\mu{}^\kappa B_j{}^\mu = v^i k_{ji} C^\kappa + (\nabla_j v^i) B_i{}^\kappa,$$

from which,

$$\nabla_j v_i = f_{ji}, \tag{1.12}$$

$$u_j = k_{ji} v^i. \tag{1.13}$$

Equations (1.10) and (1.12) imply

$$\nabla_j v_i + \nabla_i v_j = 0. \tag{1.14}$$

If we differentiate (1.13) covariantly and use (1.12), then

$$\nabla_j u_i = (\nabla_j k_{it}) v^t + k_{it} f_j{}^t. \tag{1.15}$$

Differentiating (1.6) covariantly along M^{2n} and using (1.3) and (1.11), we have

$$-v^i R_{\nu\mu\lambda}{}^{\kappa} B_i^{\nu} B_k^{\mu} B_j^{\lambda} - k_{kj} u^i B_i^{\kappa}$$
$$= (\nabla_k f_j^h) B_h^{\kappa} + k_{kh} f_j^h C^{\kappa} + (\nabla_k u_j) C^{\kappa} - u_j k_k{}^i B_i^{\kappa},$$

from which, applying B_κ^l and C_κ respectively,

(1.16) $$\nabla_k f_{ji} = -v^t R_{tkji},$$

(1.17) $$\nabla_j u_i = -v^t (\nabla_t k_{ji} - \nabla_j k_{ti}) - k_{jt} f_i^t,$$

where R_{kjih} are covariant components of induced Riemannian curvature tensor.
Equation (1.16) implies that

(1.18) $$\nabla_k f_{ji} + \nabla_j f_{ik} + \nabla_i f_{kj} = 0$$

because of first Bianchi identity.
Combining (1.15) and (1.17), we find

(1.19) $$v^t (\nabla_t k_{ji}) = -(k_{jt} f_i^t + k_{it} f_j^t),$$

or equivalently,

(1.20) $$\mathcal{L}_v k_{ji} = 0,$$

denoting $\mathcal{L}_v$ by the operator of Lie differentiation with respect to vector field v^h.
On the other hand, by the Codazzi equation of M^{2n}, that is,

$$\nabla_k k_{ji} - \nabla_j k_{ki} = R_{\lambda\mu\nu\kappa} B_k^{\lambda} B_j^{\mu} B_i^{\nu} C^{\kappa}$$

and (1.4), we obtain

(1.21) $$v^j v^i (\nabla_k k_{ji} - \nabla_j k_{ki}) = 0,$$

from which, using (1.13),

(1.22) $$(\nabla_j k_{ki}) v^j v^i = (\nabla_k k_{ji}) v^j v^i = 0.$$

We easily see, from (1.15) and the third equation of (1.19),

(1.23) $$\mathcal{L}_v u^h = 0, \qquad \mathcal{L}_u v^h = 0.$$

And we also find that

(1.24) $$\mathcal{L}_v f_i^h = 0$$

by virtue of (1.12) and (1.16).
Finally we put a tensor field of type (1, 2) as

(1.25) $$S_{ji}^h = f_j^t \nabla_t f_i^h - f_i^t \nabla_t f_j^h - (\nabla_j f_i^t - \nabla_i f_j^t) f_t^h$$
$$+ (\nabla_j u_i - \nabla_i u_j) u^h + (\nabla_j v_i - \nabla_i v_j) v^h.$$

If the tensor S_{ji}^h vanishes, then the complemented f-structure is said to be *normal* (cf. [5]). We easily verify

(1.26) $$\mathcal{L}_v S_{ji}^h = 0$$

by virtue of (1.14), (1.23) and (1.24).

2. M^{2n} as fibred space with 1-dimensional fibres.

In the sequel we consider M^{2n} with a complemented f-structure immersed in a K-contact metric manifold $\tilde{M}^{2n+1}$. We assume in the sequel that the vector field v^h is regular and defines a fibred space $\{M^{2n}, \bar{M}^{2n-1}, g, \pi\}$ with invariant Riemannian metric because of (1.14), where π is projection

(2.1) $$\pi : M^{2n} \longrightarrow \bar{M}^{2n-1},$$

$\bar{M}^{2n-1}$ being the base manifold.

The distribution whose tangent space is spanned by v^h is integrable because of (1.23). Thus we also assume that this distribution is regular and consider a fibering of M^{2n}.

Covering $\bar{M}^{2n-1}$ by a system of coordinate neighborhoods $\{V; y^a\}$ $(a, b, c, \cdots=1, 2, \cdots, 2n-1)$, we represent the projection (2.1) as

(2.2) $$y^a = y^a(x^h)$$

and put

(2.3) $$E_i^a = \partial_i y^a,$$

the rank of matrix (E_i^a) being always $2n-1$. Then (E_i^a, v_i) forms a local coframe in M^{2n}. If $(E^h{}_a, v^h)$ denotes the inverse matrix of (E_i^a, v_i), then it forms the frame corresponding to the coframe and we have

(2.4) $$E_i^a E^i{}_b = \delta_b^a, \qquad E_i^a v^i = 0, \qquad v_i E^i{}_b = 0, \qquad v_i v^i = 1,$$

from which,

(2.5) $$E_i^a E^h{}_a + v_i v^h = \delta_i^h,$$

Since we have

(2.6) $$\mathcal{L}_v E_i^a = v^j \nabla_j E_i^a + E_j^a \nabla_i v^j = \partial_i (v^j E_j^a) = 0,$$

we can see that

(2.7) $$\mathcal{L}_v E^h{}_a = 0.$$

An arbitrary vector field X^h can be written as

(2.8) $$X^h = E^h{}_a X^a + v^h X,$$

where $X^a = E_i^a X^i$, $X = v_i X^i$. If a vector field X^h satisfies $X^h = E^h{}_a X^a$, then X^h is said to be horizontal. If X^h satisfies $X^h = v^h X$, then it is said to be vertical.

For a horizontal vector field $X^h = E^h{}_a X^a$, we have $\mathcal{L}_v X^h = E^h{}_a \mathcal{L}_v X^a$. If we assume that $(\mathcal{L}_v X^h) E_h^a = 0$, then we have $\mathcal{L}_v X^a = 0$, which shows that X^h induces a vector field X^a in $\bar{M}^{2n-1}$.

The square of the length of horizontal vector field $X^h = E^h{}_a X^a$ is given by $g_{cb} X^c X^b$, where

(2.9) $$g_{cb} = E^j{}_c E^i{}_b g_{ji}.$$

Taking account of (1.14) and (2.7), we see that $\mathcal{L}_v g_{cb}=0$, which shows g_{cb} given by (2.9) defines a Riemannian metric in $\bar{M}^{2n-1}$.

From (2.12) we have

(2.10) $$g_{ji}=E_j{}^c E_i{}^b g_{cb}+v_j v_i.$$

We can easily see that $E_i{}^a$ and $E^h{}_b$ are related by $E^h{}_b=E_i{}^a g^{ih} g_{ab}$.

The transform $f_i{}^h E^i{}_b$ of $E^i{}_b$ by $f_i{}^h$ can be expressed by

$$f_i{}^h E^i{}_b=f_b{}^a E^h{}_a+p_b v^h,$$

from which, $p_b=0$ by virtue of (1.9) and (2.4) and consequently we have

(2.11) $$f_i{}^h E^i{}_b=f_b{}^a E^h{}_a.$$

Using (1.24) and (2.7), we obtain $(\mathcal{L}_v f_b{}^a)E^h{}_a=0$, from which,

(2.12) $$\mathcal{L}_v f_b{}^a=0,$$

which shows that $f_b{}^a$ defines a tensor field of type (1,1) in $\bar{M}^{2n-1}$.

Applying $f_h{}^k$ to (2.11) and taking account of (2.4), we have

(2.13) $$f_b{}^c f_c{}^a=-\delta_b{}^a+u_b u^a,$$

where

(2.14) $$u_b=E^i{}_b u_i, \qquad u^a=E_i{}^a u^i,$$

from which,

(2.15) $$u^a=u_b g^{ba},$$

where $(g_{cb})^{-1}=(g^{cb})$.

From the second equation of (2.14) and using (1.23), we have $\mathcal{L}_v u^a=0$, which shows that u^a defines a vector field in $\bar{M}^{2n-1}$.

We can see that u^h can be written in the form

$$u^h=E^h{}_a u^a+\rho v^h,$$

from which, $\rho=0$ by the virtue of (2.4), and consequently

(2.16) $$u^h=E^h{}_a u^a.$$

Applying $f_h{}^k$ to (2.16) and using (2.11), we have

(2.17) $$f_a{}^b u^a=0.$$

Similarly, we find from (1.8)

(2.18) $$g_{dc} f_b{}^d f_a{}^c=g_{ba}-u_b u_a$$

and

(2.19) $$u_a u^a=1.$$

Thus we have

THEOREM 2.1. *Let* $\{M^{2n}, \bar{M}^{2n-1}, g, \pi\}$ *be a fibred Riemannian space such that* M^{2n} *is* *differentiable manifold with complemented f-structure immersed in a K-contact metric manifold $\tilde{M}^{2n+1}$, g is the Riemannian metric tensor induced from that of $\tilde{M}^{2n+1}$, and $\pi: M^{2n}\longrightarrow \bar{M}^{2n-1}$ is a projection defined by the regular vector field v^h. Then $\bar{M}^{2n-1}$ has an almost contact metric structure.*

The van der Waerden-Borotolitti covariant derivative of $\nabla_j E^h{}_b$ and $\nabla_j E_i{}^a$ are given by

$$(2.20)\qquad \begin{aligned} \nabla_j E^h{}_b &= \partial_j E^h{}_b + \left\{{}_j{}^h{}_i\right\} E^i{}_b - \left\{{}_c{}^a{}_b\right\} E_j{}^c E^h{}_a,\\ \nabla_j E_i{}^a &= \partial_j E_i{}^a - \left\{{}_j{}^h{}_i\right\} E_h{}^a + \left\{{}_c{}^a{}_b\right\} E_j{}^c E_i{}^b, \end{aligned}$$

where $\left\{{}_c{}^a{}_b\right\}$ are the Christoffel symbols formed with g_{cb}.

The co-Gauss and co-Weingarten equations with respect to this covariant derivatives are given by

$$(2.21)\qquad \nabla_j E^h{}_b = h_{cb} E_j{}^c v^h - h_b{}^a v_j E^h{}_a, \qquad \nabla_j v^h = -h_c{}^a E_j{}^c E^h{}_a$$

and

$$(2.22)\qquad \nabla_j E_i{}^a = h_c{}^a (E_j{}^c v_i + v_j E_i{}^c), \qquad \nabla_j v_i = -h_{cb} E_j{}^c E_i{}^b,$$

where h_{cb} is a 2-form of $\bar{M}^{2n-1}$ and $h_c{}^a = h_{cb} g^{ba}$.

Equations (1.12) and (2.22) imply

$$(2.23)\qquad h_{cb} = -f_{cb}.$$

Thus (2.21) and (2.22) become respectively

$$(2.24)\qquad \nabla_j E^h{}_b = f_{bc} E_j{}^c v^h + f_b{}^a v_j E^h{}_a, \qquad \nabla_j v^h = f_c{}^a E_j{}^c E^h{}_a$$

and

$$(2.25)\qquad \nabla_j E_i{}^a = -f_c{}^a (E_j{}^c v_i + v_j E_i{}^c), \qquad \nabla_j v_i = f_{cb} E_j{}^c E_i{}^b.$$

Applying the operator $\nabla_c = E^j{}_c \nabla_j$ to $f_b{}^a = f_i{}^h E^i{}_b E_h{}^a$, we can find

$$(2.26)\qquad \nabla_c f_b{}^a = E^j{}_c (\nabla_j f_i{}^h) E^i{}_b E_h{}^a$$

because of (2.24), (2.25), $f_i{}^h v^i = v_h f_i{}^h = 0$ and $v_h E^h{}_c = 0$.

From (2.26) we have

$$(2.27)\qquad \nabla_c f_{ba} + \nabla_b f_{ac} + \nabla_a f_{cb} = 0$$

because of (1.18).

Thus we conclude

PROPOSITION 2.2. *The base manifold of a fibred space as those stated in theorem 2.1 has a closed 2-form.*

Substituting (1.16) into (2.26), we find

$$(2.28)\qquad \nabla_c f_b{}^a = R_{cob}{}^a,$$

where $R_{cob}{}^a = R_{jti}{}^h E^j{}_c v^t E^i{}_b E_h{}^a$ denote the covariant components of curvature tensor of the induced connection ∇_c (cf. [3], [4]).

We compute a tensor field of type (1, 3) such that

$$S_{cb}{}^a = f_c{}^e \nabla_e f_b{}^a - f_b{}^e \nabla_e f_c{}^a - (\nabla_c f_b{}^e - \nabla_b f_c{}^e) f_e{}^a + (\nabla_c u_b - \nabla_b u_c) u^a. \tag{2.29}$$

Substituting (2.28) into this equation, we find

$$S_{cb}{}^a = f_c{}^e R_{eob}{}^a - f_b{}^e R_{eoc}{}^a - (R_{cob}{}^e - R_{boc}{}^e) f_e{}^a + (\nabla_c u_b - \nabla_b u_c) u^a. \tag{2.30}$$

If we define a tensor field of type (1, 2) such that

$$\tilde{S}_{cb}{}^a = S_{ji}{}^h E^j{}_c E^i{}_b E_h{}^a,$$

then we have $\mathcal{L}_v \tilde{S}_{cb}{}^a = 0$ because of (1.26), which shows that $\tilde{S}_{cb}{}^a$ is an induced tensor field in $\bar{M}^{2n-1}$.

Finally we can prove that $\tilde{S}_{cb}{}^a = S_{cb}{}^a$. In fact,

$$\begin{aligned} & S_{ji}{}^h E^j{}_c E^i{}_b E_h{}^a \\ &= [-f_j{}^t v^s R_{sti}{}^h + f_i{}^t v^s R_{stj}{}^h + v^s (R_{sji}{}^t - R_{sij}{}^t) f_t{}^h \\ &\quad + (\nabla_j u_i - \nabla_i u_j) u^h + (\nabla_j v_i - \nabla_i v_j) v^h] E^j{}_c E^i{}_b E_h{}^a \\ &= f_c{}^e R_{eob}{}^a - f_b{}^e R_{eoc}{}^a - (R_{cob}{}^e - R_{boc}{}^e) f_e{}^a \\ &\quad + (\nabla_c u_b - \nabla_b u_c) u^a \end{aligned} \tag{2.31}$$

by virtue of (1.16), (2.11), (2.16) and (2.28).

Thus we have

THEOREM 2.3. *Let* $\{M^{2n}, \bar{M}^{2n-1}, g, \pi\}$ *be a fibred space as those stated in theorem 2.1. If* M^{2n} *is normal, so is* $\bar{M}^{2n-1}$.

3. M^{2n} as a fibred space with 2-dimensional fibres.

Let $\bar{M}^{2n-2}$ be a $(2n-2)$-dimensional differentiable manifold and denote by π the projection from M^{2n}, as a fibred space, to the base manifold $\bar{M}^{2n-2}$. And let the fibre is a 2-dimensional submanifold of M^{2n} whose tangent space is spanned by u^h and v^h.

Covering $\bar{M}^{2n-2}$ by a system of coordinate neighborhoods $\{V\,;\, y^a\}$ $(a, b, c, \cdots = 1, 2, \cdots, 2n-2)$, we can represent the projection $\pi : M^{2n} \longrightarrow \bar{M}^{2n-2}$ by

$$y^a = y^a(x^h) \tag{3.1}$$

and put

$$E_i{}^a = \partial_i y^a = (\partial/\partial x^i) y^a, \tag{3.2}$$

the rank of matrix $(E_i{}^a)$ being always $2n-2$. Then a matrix $(E_i{}^a, u_i, v_i)$ forms a local coframe in M^{2n}. If $(E^h{}_b, u^h, v^h)$ denotes the inverse matrix of $(E_i{}^a, u_i, v_i)$, then it forms a local frame and

$$\begin{aligned} & E_i{}^a E^i{}_b = \delta_b^a, \qquad E_i{}^a u^i = 0, \qquad E_i{}^a v^i = 0, \\ & u_i E^i{}_b = 0, \qquad u_i u^i = 1, \qquad u_i v^i = 0, \end{aligned} \tag{3.3}$$

$$v_iE^i{}_b=0, \qquad v_iu^i=0, \qquad v_iv^i=1,$$

from which,

(3.4) $$E_i{}^aE^h{}_a+u_iu^h+v_iv^h=\delta_i^h.$$

Since we have

(3.5) $$\mathcal{L}_vE_i{}^a=0, \qquad \mathcal{L}_uE_i{}^a=0,$$

then we see

(3.6) $$\mathcal{L}_vE^h{}_a=0, \qquad \mathcal{L}_uE^h{}_a=0$$

because of (1.22) and (1.23).

Equation (3.4) shows that an arbitrary vector field X^h can be written as

(3.7) $$X^h=E^h{}_aX^a+v^hX+u^hX',$$

where $X^a=E_i{}^aX^i$, $X=v_iX^i$, $X'=u_iX^i$. If a vector field X^h satisfies $X^h=E^h{}_aX^a$, then X^h is said to be horizontal. If X^h satisfies $X^h=v^hX+u^hX'$, then X^h is said to be vertical.

For a horizontal vector field $X^h=E^h{}_aX^a$ we have

(3.8) $$\mathcal{L}_vX^h=E^h{}_a\mathcal{L}_vX^a, \quad \mathcal{L}_uX^h=E^h{}_a\mathcal{L}_uX^a.$$

If we assume that $(\mathcal{L}_vX^h)E_h{}^a=0$, $(\mathcal{L}_uX^h)E_h{}^a=0$, then we have $\mathcal{L}_vX^a=0$, $\mathcal{L}_uX^a=0$, and consequently X^h induce a vector field X^a in $\bar{M}^{2n-2}$.

From (1.14) we see that v^h is a Killing vector field. We assume that, in this section, u^h is a Killing vector field. Taking account of (1.15) and the fact that u^h is Killing, we have $g_{hk}\mathcal{L}_uf_i{}^h=0$, from which,

(3.9) $$\mathcal{L}_uf_i{}^h=0.$$

The square of the length of horizontal vector $X^h=E^h{}_aX^a$ is given by $g_{cb}X^cX^b$, where

(3.10) $$g_{cb}=E^j{}_cE^i{}_bg_{ji}.$$

Using (3.6) and the fact that u^h, v^h are Killing, we have $\mathcal{L}_vg_{cb}=0$, $\mathcal{L}_ug_{cb}=0$, which mean that g_{cb} given by (3.10) define a Riemannian metric in $\bar{M}^{2n-2}$. Equation (3.4) is rewritten as

(3.11) $$g_{ji}=E_j{}^cE_i{}^bg_{cb}+u_ju_i+v_jv_i.$$

The transform $f_i{}^hE^i{}_b$ of $E^i{}_b$ by $f_i{}^h$ can be expressed by

$$f_i{}^hE^i{}_b=f_b{}^aE^h{}_a+p_bv^h+q_bu^h,$$

from which, $p_b=q_b=0$ because of (1.9) and (3.3), and consequently

(3.12) $$f_i{}^hE^i{}_b=f_b{}^aE^h{}_a.$$

If we take account of (1.24), (3.6) and (3.9), then we have

(3.13) $$\mathcal{L}_vf_b{}^a=0, \qquad \mathcal{L}_uf_b{}^a=0,$$

which imply that $f_b{}^a$ define a tensor field of type (1, 1) in $\tilde{M}^{2n-2}$.

Applying $f_h{}^k$ to (3.12), we have

$$f_b{}^c f_c{}^a = -\delta_b{}^a. \tag{3.14}$$

Applying also $g_{th} f_j{}^t E^j{}_c$ to (3.12), we have

$$g_{ed} f_c{}^e f_b{}^d = g_{cb}. \tag{3.15}$$

Thus we have

THEOREM 3.1. *Let $\{M^{2n}, \tilde{M}^{2n-2}, g, \pi\}$ be a fibred Riemannian space such that M^{2n} is differentiable manifold with complemented f-structure immersed in a K-contact metric manifold $\tilde{M}^{2n+1}$, g is the Riemannian metric tensor induced from that of $\tilde{M}^{2n+1}$, and $\pi : M^{2n} \longrightarrow \tilde{M}^{2n-2}$ is a projection defined by the regular vector field u^h and v^h. If u^h is a Killing, then $\tilde{M}^{2n-2}$ is an almost Hermitian manifold.*

If $\nabla_j E_i{}^a$ and $\nabla_j E^h{}_b$ are defined by (2.20), then the equations of co-Gauss and those of co-Weingarten are given by

$$\begin{aligned} \nabla_j E_i{}^a &= h_c{}^a (E_j{}^c v_i + v_j E_i{}^c) + H_c{}^a (E_j{}^c u_i + u_j E_i{}^c), \\ \nabla_j v_i &= -h_{cb} E_j{}^c E_i{}^b, \qquad \nabla_j u_i = -H_{cb} E_j{}^c E_i{}^b \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \nabla_j E^h{}_b &= h_{cb} E_j{}^c v^h - h_b{}^a v_j E^h{}_a + H_{cb} E_j{}^c u^h - H_b{}^a u_j E^h{}_a, \\ \nabla_j v^h &= -h_c{}^a E_j{}^c E^h{}_a, \quad \nabla_j u^h = -H_c{}^a E_j{}^c E^h{}_a, \end{aligned} \tag{3.17}$$

where h_{cb} and H_{cb} are 2-forms of $\tilde{M}^{2n-2}$ and $h_c{}^a = h_{cb} g^{ba}$, $H_c{}^a = H_{cb} g^{ba}$.

From (1.12) and (3.16) we find

$$h_{cb} = -f_{cb}. \tag{3.18}$$

Applying the operator $\nabla_c = E^j{}_c \nabla_j$ to $f_b{}^a = f_i{}^h E^i{}_b E_h{}^a$, we have

$$\nabla_c f_b{}^a = E^j{}_c (\nabla_j f_i{}^h) E^i{}_b E_h{}^a \tag{3.19}$$

by virtue of (3.3), (3.16), (3.17), (3.18) and $u_t f_j{}^t = v_t f_j{}^t = 0$. From (1.18) and (3.19)

$$\nabla_c f_{ba} + \nabla_b f_{ac} + \nabla_a f_{cb} = 0, \tag{3.20}$$

that is, $f_c{}^a$ is an almost Kaehlerian structure (cf. [2]).

Thus we have

THEOREM 3.2. *The base manifold of a fibred space as those stated in theorem 3.1 is an almost Kaehlerian manifold.*

Now we define a tensor field of type (1, 2) such that

$$\tilde{S}_{cb}{}^a = S_{ji}{}^h E^j{}_c E^i{}_b E_h{}^a.$$

Then, taking account of (1.23), (1.24), (1.25) and (3.9), we have

(3.22) $$\mathcal{L}_v S_{ji}{}^h=0, \qquad \mathcal{L}_u S_{ji}{}^h=0,$$

from which,

(3.23) $$\mathcal{L}_v \tilde{S}_{cb}{}^a=0, \qquad \mathcal{L}_u \tilde{S}_{cb}{}^a=0$$

by virtue of (3.5), (3.6). Equations (3.23) shows that $\tilde{S}_{cb}{}^a$ is an induced tensor field in $\bar{M}^{2n-1}$.

But we can see that $\tilde{S}_{cb}{}^a=S_{cb}{}^a$ because of (3.3) and (3.9). And consequently if $S_{ji}{}^h=0$, then $S_{cb}{}^a=0$, that is, $f_c{}^a$ is a Kaehlerian structure (cf. [2]).

Thus we have

THEOREM 3.3. *Let $\{M^{2n}, \bar{M}^{2n-2}, g, \pi\}$ be a fibred space as those stated in theorem 3.1. If M^{2n} is normal, then $\bar{M}^{2n-2}$ is a Kaehlerian manifold.*

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