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A SHORT PROOF OF BARBUT'S THEOREM

BY HYUNKOO LEE

Erol Barbut [1] defines the Levitzki radical $L(R)$ of a semiring and shows that every nil subsemiring of a semiring R with the ascending chain conditions on left and right annihilator ideals is nilpotent, provided that $L(R)$ is a k-ideal. A left (right) ideal I of a semiring R is called a left(right) *k-ideal* if $x+y\in I$ and $y\in I$ implies that $x\in I$ for each x and y in R. A left (right) ideal I of a semiring R is a *left (right) annihilator* if there exists a subset S of R such that $I=l(S)=\{x\in R\mid xS=(0)\}$, $I=r(S)=\{x\in R\mid Sx=(0)\}$.

In this paper, we give a short proof of Barbut's theorem, that is, every nil subsemiring of a semiring with the ascending chain conditions on left and right annihilators is nilpotent.

A semiring R is said to be *left T-nilpotent* if for each sequence $\{x_n\}$ of elements in R there exists an n such that $x_1x_2\cdots x_n=0$.

LEMMA (cf. [2]). *Let a semiring R satisfy the ascending chain condition on left annihilators. Then a subsemiring S of R is nilpotent if and only if it is left T-nilpotent.*

Proof. Suppose that S is left T-nilpotent. Since the ascending chain condition of left annihilators is inherited from the semiring by its subsemirings there exists m such that

$$l(S^m)=l(S^{m+t}) \text{ for all } t\geqq 1.$$

If $S^{m+1}\neq(0)$, then there exists $x_1\in S$ such that $x_1S^m\neq(0)$. Then $x_1S^{m+1}\neq(0)$ and hence there exists $x_2\in S$ such that $x_1x_2S^m\neq(0)$. Then $x_1x_2S^{m+1}\neq(0)$ and there exists $x_3\in S$ such that $x_1x_2x_3S^m\neq(0)$. Continuing in this manner, we obtain a sequence $\{x_n\}$ in S such that $x_1x_2\cdots x_n\neq 0$ for each n. This contradicts the left T-nilpotency of S. Therefore S is nilpotent. The proof of the converse is obvious.

THEOREM (cf. [3]). *If R is a semiring satisfying the ascending chain conditions on left and right annihilators, then any nil subsemiring of R is nilpotent.*

Proof. Since the ascending chain conditions on left and right annihilators are inherited from the semiring by its subsemirings, we may, without loss of generality, assume that R is nil. We wish to show that R is nilpotent.

We say that $x_1\in R$ has an infinite chain if there exists an infinite sequence $\{x_n\}$ such that $x_1x_2\cdots x_n\neq 0$ for all n.

Suppose that R is not left T-nilpotent. Then there exist elements which have an infinite chain. Let $l(x_0)$ be maximal in $\{l(x)\mid x \textit{ has an infinite chain}\}$. And let $l(x_1)$ be maximal in $\{l(x)\mid x_0x \textit{ has an infinite chain}\}$. Inductively we find x_n such that $l(x_n)$ is maximal in $\{l(x)\mid x_0x_1x_2\cdots x_{n-1}\, x \textit{ has an infinite chain}\}$. It is easy to see that

$$l(x_k)=l(x_kx_{k+1}x_{k+2}\cdots x_{k+t}) \text{ for all } k \text{ and } t.$$

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We claim that $x_0x_1x_2\cdots x_nx_k=0$ for each n and $k\leqq n$. If $x_0x_1x_2\cdots x_nx_k\neq 0$ for some n and $k\leqq n$, then $x_0x_1x_2\cdots x_nx_kx_{k+1}\cdots x_{k+t}\neq 0$ for all t. Hence $x_0x_1x_2\cdots x_nx_k$ has an infinite chain. Whence $l(x_kx_{k+1}\cdots x_nx_k)=l(x_k)$. However, this is impossible since $x_kx_{k+1}\cdots x_n$ is a nilpotent element. Therefore $x_0x_1x_2\cdots x_nx_k=0$ for each n and $k\leqq n$. Let $y_k=x_0x_1x_2\cdots x_k$ for each k and let $S_i=\{y_k\,|\,k\geqq i\}$, $i=1,2,3,\cdots$ Then $r(S_k)$ is properly contained in $r(S_{k+1})$ for all k since $y_kx_{k+1}\neq 0$ for all k and yet $y_nx_{k+1}=0$ for all k and $n\geqq k+1$. This is a contradiction. Therefore R is left T-nilpotent. Hence, by the previous lemma, R is nilpotent.

COROLLARY 1 (Erol Barbut). *If R is a semiring which satisfies the ascending chain conditions on left and right annihilators and is such that $L(R)$ is a k-ideal, then any nil subsemiring of R is nilpotent.*

The proof is evident.

COROLLARY 2. *If R is a semiring satisfying the ascending chain conditions on left and right k-ideals, then any nil subsemiring of R is nilpotent.*

Proof. Since every left or right annihilator is a left or right k-ideal, the result follows immediately from the theorem.

References

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[3] I. H. Herstein and Lance Small, *Nil rings satisfying certain chain conditions*, Can. J. Math. **16** (1964), 771–776.

Seoul National University