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AN INVARIANT OF COMPLEX CONFORMAL CONNECTIONS IN A KÄHLER MANIFOLD

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0. Preliminaries.

We consider an n-dimensional Kähler manifold M covered by a system of coordinate neighborhoods $\{U : \xi^h\}$, and denote by g_{ij} and F_i^h the components of the Hermitian metric tensor and the complex structure tensor of M respectively, where the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, n\}$.

We denote by ∇_j the operator of covariant differentiation with respect to the Christoffel symbols Γ_{ji}^h (instead of $\{{}_j{}^h{}_i\}$) formed with g_{ji}, then we have

$$\nabla_k g_{ji}=0, \qquad \nabla_k F_i^h=0, \qquad \nabla_k F_{ji}=0,$$

where $F_{ji}=F_j^t g_{ti}$ and consequently $F_{ji}=-F_{ij}$.

And also, we denote by

$$R_{kji}{}^h=\partial_k \Gamma_{ji}^h-\partial_j \Gamma_{ki}^h+\Gamma_{kt}^h \Gamma_{ji}^t-\Gamma_{jt}^h \Gamma_{ki}^t,$$

where $\partial_k=\partial/\partial\xi^k$, the components of the Riemann-Christoffel curvature tensor of M, by $R_{ji}=R_{tji}{}^t$ the Ricci tensor and by $R=R_{ji}g^{ji}$ the scalar curvature.

1. Introduction.

We consider an affine connection in a Kähler manifold M, and denote by $\bar{\Gamma}_{ji}^h$ the components of the connection and by $\bar{\nabla}_j$ the operator of covariant differentiation with respect to $\bar{\Gamma}_{ji}^h$.

Now, we consider a conformal change of Hermitian metric

$$\bar{g}_{ji}=e^{2p}g_{ji}, \qquad \bar{F}_i^h=F_i^h, \qquad \bar{F}_{ji}=e^{2p}F_{ji}, \tag{1}$$

where p is a scalar function and we look for an affine connection such that

$$\bar{\nabla}_k \bar{g}_{ji}=0, \qquad \bar{\nabla}_k \bar{F}_{ji}=0, \tag{2}$$

and the torsion tensor $S_{ji}{}^h=\frac{1}{2}(\bar{\Gamma}_{ji}^h-\bar{\Gamma}_{ij}^h)$ is given by

$$S_{ji}{}^h=-F_{ji}q^h, \tag{3}$$

where q^h are components of a vector field. Then, the components $\bar{\Gamma}_{ji}^h$ of this affine connection are given by

$$\bar{\Gamma}_{ji}^h=\Gamma_{ji}^h+\delta_j^h p_i+\delta_i^h p_j-g_{ji}p^h+F_j^h q_i+F_i^h q_j-F_{ji}q^h, \tag{4}$$

where $p_i=\partial_i p$, $p^h=p_t g^{th}$, $q_i=-p_t F_i^t$ and $q^h=q_t g^{th}$,
According to K. Yano [5], this connection is called a complex conformal connection,

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and the curvature tensor of $\bar{\Gamma}_{ji}^h$ is given by

$$\bar{R}_{kji}{}^h = R_{kji}{}^h - \delta_k^h p_{ji} + \delta_j^h p_{ki} - p_k{}^h g_{ji} + p_j{}^h g_{ki} - F_k{}^h q_{ji} + F_j{}^h q_{ki} - q_k{}^h F_{ji} + q_j{}^h F_{ki} - \alpha_{kj} F_i^h - F_{kj} \beta_i^h, \tag{5}$$

where

$$p_{ji} = \nabla_j p_i - p_j p_i + q_j q_i + \frac{1}{2} p_t p^t g_{ji} \tag{6}$$

$$q_{ji} = \nabla_j q_i - p_j q_i - q_j p_i + \frac{1}{2} p_t p^t F_{ji} \tag{7}$$

$$\alpha_{kj} = -(\nabla_k q_j - \nabla_j q_k) = q_{jk} - q_{kj} + p_t p^t F_{kj}, \tag{8}$$

$$\beta_i^h = 2(p_i q^h - q_i p^h), \tag{9}$$

$p_k{}^h = p_{kt} g^{th}$, $q_k{}^h = q_{kt} g^{th}$ and consequently

$$q_{ji} = -p_{jt} F_i^t, \qquad p_{ji} = q_{jt} F_i^t. \tag{10}$$

K. Yano [5] studied on complex conformal connection and he proved the following

THEOREM. *If, in an n-dimensional Kähler manifold* ($n \geqq 4$), *there exists a scalar function p such that the complex conformal connection is of zero curvature, then the Bochner curvature tensor of the manifold vanishes.*

In the present paper, we find an invariant curvature tensor of the complex conformal connection, and generalize the above theorem.

2. An invariant curvature tensor.

In (5), we contract with respect to h and k, and use $F_k{}^k=0$, $q_k{}^k=0$, then we obtain

$$\bar{R}_{ji} = R_{ji} - (n-2) p_{ji} - P g_{ji} + q_{ti} F_j{}^t - q_{jt} F_i^t - \alpha_{tj} F_i^t - F_{tj} \beta_i^t, \tag{11}$$

where $P = p_t^t$, and transvecting with $g^{ji} (= e^{2p} \bar{g}^{ji})$, we have

$$e^{2p} \bar{R} = R - 2(n+1) P + (n+4) \lambda, \tag{12}$$

where $\lambda = p_t p^t$

If we define V_{kj} by $2V_{kj} = R_{kjs}{}^t F_t^s$, from (5), we have

$$\bar{V}_{kj} = V_{kj} + \frac{n+4}{2} \alpha_{kj} \tag{13}$$

and transvecting with $F^{kj} (= e^{2p} \bar{F}^{kj})$, we have

$$e^{2p} \bar{V} = V + \frac{n+4}{2} \alpha \tag{14}$$

$$=V-(n+4)P+\frac{1}{2}n(n+4)\lambda,$$

where $V=V_{kj}F^{kj}$, $\alpha=\alpha_{kj}F^{kj}$.

From (12) and (14), we find

$$\lambda=\frac{1}{n^2-4}\left\{\frac{2(n+1)}{n+4}\bar{V}-\bar{R}\right\}e^{2p}-\frac{1}{n^2-4}\left\{\frac{2(n+1)}{n+4}V-R\right\} \tag{15}$$

$$P=\frac{1}{n^2-4}\left(\bar{V}-\frac{n}{2}\bar{R}\right)e^{2p}-\frac{1}{n^2-4}\left(V-\frac{n}{2}R\right) \tag{16}$$

If we define H_{jk} by $H_{jk}=-R_{jt}F_k{}^t$, then we obtain

$$\bar{H}_{jk}=H_{jk}-nq_{jk}+2q_{kj}-(P-\lambda)F_{jk}+\beta_{jk}, \tag{17}$$

from which

$$\bar{H}_{jk}+\bar{H}_{kj}=H_{jk}+H_{kj}-(n-2)(q_{jk}+q_{kj}). \tag{18}$$

Therefore, we have

$$q_{jk}+q_{kj}=\frac{1}{n-2}(H_{jk}+H_{kj})-\frac{1}{n-2}(\bar{H}_{jk}+\bar{H}_{kj}). \tag{19}$$

On the other hand, from (13) and using (8), we have

$$q_{jk}-q_{kj}=\frac{2}{n+4}(\bar{V}_{kj}-V_{kj})-\lambda F_{kj}. \tag{20}$$

From (19) and (20), and using (15), we find

$$\begin{aligned} q_{jk}=&\frac{1}{2(n-2)}(H_{jk}+H_{kj})-\frac{1}{n+4}V_{kj}+\frac{1}{n^2-4}\left(\frac{n+1}{n+4}V-\frac{1}{2}R\right)F_{kj} \\ &-\frac{1}{2(n-2)}(\bar{H}_{jk}+\bar{H}_{kj})+\frac{1}{n+4}\bar{V}_{kj}-\frac{1}{n^2-4}\left(\frac{n+1}{n+4}\bar{V}-\frac{1}{2}\bar{R}\right)\bar{F}_{kj}. \end{aligned} \tag{21}$$

If we define M_{jk} by

$$M_{jk}=-\frac{1}{2(n-2)}(H_{jk}+H_{kj})+\frac{1}{n+4}V_{kj}-\frac{1}{n^2-4}\left(\frac{n+1}{n+4}V-\frac{1}{2}R\right)F_{kj}, \tag{22}$$

then (21) can be written as

$$q_{jk}=\bar{M}_{jk}-M_{jk}. \tag{23}$$

Therefore, using (10), we have

$$p_{ji}=\bar{M}_{jt}\bar{F}_i{}^t-M_{jt}F_i{}^t$$

and writing L_{ji} as

$$L_{ji}=M_{jt}F_i{}^t, \tag{24}$$

we have

$$p_{ji}=\bar{L}_{ji}-L_{ji}. \tag{25}$$

Substituting (15), (16) and (23) into (17), we obtain

$$\beta_{jk}=\bar{H}_{jk}+n\bar{M}_{jk}-2\bar{M}_{kj}-\frac{1}{n+2}\left(\frac{1}{n+4}\bar{V}+\frac{1}{2}\bar{R}\right)\bar{F}_{jk} \tag{26}$$
$$-H_{jk}-nM_{jk}+2M_{kj}+\frac{1}{n+2}\left(\frac{1}{n+4}V+\frac{1}{2}R\right)F_{jk}.$$

And hence, if we define B_{jk} by

$$B_{jk}=H_{jk}+nM_{jk}-2M_{kj}-\frac{1}{n+2}\left(\frac{1}{n+4}V+\frac{1}{2}R\right)F_{jk}, \tag{27}$$

we have

$$\beta_{jk}=\bar{B}_{jk}-B_{jk}. \tag{28}$$

Finally, substituting (13), (23), (25) and (28) into (5),

we have

$$\bar{R}_{kji}{}^h+\delta_k^h\bar{L}_{ji}-\delta_i^h\bar{L}_{ki}+\bar{L}_k^h\bar{g}_{ji}-\bar{L}_j^h\bar{g}_{ki}+\bar{F}_k{}^h\bar{M}_{ji}-\bar{F}_j{}^h\bar{M}_{ki}$$
$$+\bar{M}_k^h\bar{F}_{ji}-\bar{M}_j^h\bar{F}_{ki}+\frac{2}{n+4}\bar{V}_{kj}\bar{F}_i{}^h+\bar{F}_{kj}\bar{B}_i^h$$
$$=R_{kji}{}^h+\delta_k^hL_{ji}-\delta_j^hL_{ki}+L_k^hg_{ji}-L_j^hg_{ki} \tag{29}$$
$$+F_k{}^hM_{ji}-F_j{}^hM_{ki}$$
$$+M_k^hF_{ji}-M_j^hF_{ki}$$
$$+\frac{2}{n+4}V_{kj}F_i{}^h+F_{kj}B_i^h,$$

where $L_k^h=L_{kt}g^{th}$, $M_k^h=M_{kt}g^{th}$ and $B_i^h=B_{it}g^{th}$.

Defining $C_{kji}{}^h$ as

$$C_{kji}{}^h=R_{kji}{}^h+\delta_k^hL_{ji}-\delta_j^hL_{ki}+L_k^hg_{ji}-L_j^hg_{ki} \tag{30}$$
$$+F_k{}^hM_{ji}-F_j{}^hM_{ki}+M_k^hF_{ji}-M_j^hF_{ki}$$
$$+\frac{2}{n+4}V_{kj}F_i{}^h+F_{kj}B_i^h,$$

(29) reduces into

$$\bar{C}_{kji}{}^h=C_{kji}{}^h. \tag{31}$$

We call such a tensor $C_{kji}{}^h$ a *complex conformal curvature tensor*. Thus, we have the following theorem.

THEOREM 1. *In an n-dimensional Kähler manifold* ($n\geqq4$), *the complex conformal curvature tensor* $C_{kji}{}^h$ *defined by* (30) *is invariant under a change of the complex conformal connection defined by* (4).

By the properties of the Riemann-Christoffel curvature tensors $R_{kji}{}^h$ and $R_{kjih}=R_{kji}{}^t g_{th}$, we can easily see that

$$V_{kj}=\frac{1}{2}R_{kjs}{}^tF_t{}^s=-\frac{1}{2}R_{kjst}F^{st}=H_{kj},$$

$$V=V_{kj}F^{kj}=H_{kj}F^{kj}=R.$$

Therefore, $H_{jk}+H_{kj}=0$, and also

$$M_{jk}=-\frac{1}{n+4}H_{jk}+\frac{1}{2(n+2)(n+4)}RF_{jk},$$

$$B_{jk}=-\frac{2}{n+4}H_{kj}+\frac{2}{(n+2)(n+4)}RF_{kj},$$

hence, we have

$$\frac{2}{n+4}V_{kj}F_i{}^h+F_{kj}B_i{}^h$$
$$=-2(M_{kj}F_i{}^h+F_{kj}M_i{}^h)$$

That is, the complex conformal curvature tensor $C_{kji}{}^h$ of the Riemann-Christoffel symbols Γ_{ji}^h in a Kähler manifold is nothing but the Bochner curvature tensor, although $\bar{C}_{kji}{}^h$ of the complex conformal connection $\bar{\Gamma}_{ji}$ is different from Bochner curvature. Therefore the Yano's theorem can be extended as follows.

THEOREM 2. *In an n-dimensional Kähler manifold* $(n\geqq 4)$, *a necessary and sufficient condition that the complex conformal curvature tensor of the complex conformal connection to be of zero curvature is that the Bochner curvature tensor of the manifold vanishes.*

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