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A NOTE ON COMPLEX CONFORMAL CONNECTIONS

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1. Introduction.

K. Yano introduced what he call complex conformal connection and studied the condition for a Kaehler manifold to admit a complex conformal connection whose curvature tensor vanishes. In [1], K. Yano proved the following

THEOREM *If, in an n-dimensional Kaehler manifold* ($n \geqq 4$), *there exists a scalar function p such that the complex conformal connection is of zero curvature, then the Bochner curvature tensor of the manifold vanishes.*

In this paper we investigate a relation between the Bochner curvature tensor and the curvature tensor of the complex conformal connection and we obtain a condition, under the some assumptions, for the Bochner curvature tensor of a Kaehler manifold to be zero.

2. Preliminaries.

Let M be an n-dimensional Kaehler manifold covered by a system of coordinate neighborhoods $\{U ; x^k\}$, where here and in the sequel the indices $k, j, i, \cdots$ run over the range $\{1, 2, \cdots\cdots, n\}$ and denote by g_{ji} and $F_i{}^h$ the components of the Hermitian metric tensor and those of the complex structure tensor of M respectively. Then we have

$$F_t{}^j F_i{}^t = -\delta_i^j, \qquad g_{st} F_j{}^s F_i{}^t = g_{ji}, \qquad \nabla_k F_i{}^h = 0,$$

where we denote by ∇_k the operator of covariant differentiation with respect to the Christoffel symbols $\{{}_j{}^h{}_i\}$ formed with g_{ji}. We denote by $K_{kji}{}^h$, K_{ji} and K the Riemann-Christoffel curvature tensor, the Ricci tensor and the scalar curvature of M respectively. Then we have

$$(2.1) \qquad K_{kjih} = -K_{jkih} = -K_{kjhi}, \qquad K_{kjih} = K_{ihkj}.$$

The Bochner curvature tensor [2, 3] is defined by

$$B_{kji}{}^h = K_{kji}{}^h + \delta_k^h L_{ji} - \delta_j^h L_{ki} + L_k{}^h g_{ji} - L_j{}^h g_{ki} + F_k{}^h M_{ji} - F_j{}^h M_{ki}$$
$$+ M_k{}^h F_{ji} - M_j{}^h F_{ki} - 2(M_{kj} F_i{}^h + F_{kj} M_i{}^h),$$

where

$$L_{ji} = -\frac{1}{n+4} K_{ji} + \frac{1}{2(n+4)(n+2)} K g_{ji}, \qquad M_{ji} = -L_{jt} F_i{}^t.$$

In a Kaehler manifold, we now consider a conformal change of Hermitian metric

$$\bar{g}_{ji} = e^{2p} g_{ji}, \qquad \bar{F}_i{}^h = F_i{}^h, \qquad \bar{F}_{ji} = e^{2p} F_{ji},$$

where p is a scalar function.

The affine connection which satisfies

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$$D_k\bar{g}_{ji}=0,\ D_k\bar{F}_i^{\ h}=0 \text{ and } \Gamma_{ji}^{\ h}-\Gamma_{ij}^{\ h}=-2F_{ji}q^h,$$

where q^h is a vector field and D_k is the operator of covariant differentiation with respect to $\Gamma_{ji}^{\ h}$, is given by

(2.2) $$\Gamma_{ji}^{\ h}=\{{}_j{}^h{}_i\}+\delta_j^h p_i+\delta_i^h p_j-g_{ji}p^h+F_j^{\ h}q_i+F_i^{\ h}q_j-F_{ji}q^h,$$

where $p_i=\dfrac{\partial}{\partial x^i}p$, $\quad p^h=p_t g^{th}$, $\quad q_i=-p_t F_i^{\ t}$, $\quad q^h=q_t g^{th}$.

K. Yano [1] called such an affine connection a complex conformal connection.

3. Curvature tensor of a complex conformal connection and the Bochner curvature tensor.

We consider a complex conformal connection $\Gamma_{ji}^{\ h}$ of the equation (2.2) in a Kaehler manifold. We can find the curvature tensor $R_{kji}^{\ \ \ h}$ of $\Gamma_{ji}^{\ h}$, that is,

(3.1) $$R_{kji}^{\ \ \ h}=K_{kji}^{\ \ \ h}-\delta_k^h P_{ji}+\delta_j^h P_{ki}-P_k^{\ h}g_{ji}+P_j^{\ h}g_{ki}-F_k^{\ h}Q_{ji}+F_j^{\ h}Q_{ki}$$
$$-Q_k^{\ h}F_{ji}+Q_j^{\ h}F_{ki}+(\nabla_k q_j-\nabla_j q_k)F_i^{\ h}-2F_{kj}(p_i q^h-q_i p^h),$$

where

$$P_{ji}=\nabla_j p_i-p_j p_i+q_j q_i+(1/2)p_t p^t g_{ji},$$

$$Q_{ji}=\nabla_j q_i-p_j q_i-q_j p_i+(1/2)p_t p^t F_{ji},$$

consequently

(3.2) $$Q_{ji}=-P_{jt}F_i^{\ t},\quad P_{ji}=Q_{jt}F_i^{\ t}.$$

Therefore we have

$$R_{kjih}=-R_{kjhi} \textit{ and } R_{kji}^{\ \ \ h}=R_{jki}^{\ \ \ h},$$

where $R_{kjih}=g_{ht}R_{kji}^{\ \ \ t}$.

By a straightforward computation, we find, using (3.1),

(3.3) $$R_{kjih}+R_{jikh}+R_{ikjh}=2\{F_{ij}(p_t p^t F_{kh}+\nabla_k q_h-2q_k p_h)$$
$$+F_{ki}(p_t p^t F_{jh}+\nabla_j q_h-2q_j p_h)+F_{jk}(p_t p^t F_{ih}+\nabla_i q_h-2q_i p_h)\}.$$

If we assume that the left hand side of the equation (3.3) is zero, we have

(3.4) $$p_t p^t F_{kh}+\nabla_k q_h-2q_k p_h=0.$$

LEMMA 1. *In a Kaehler manifold the following two conditions are equivalent:*

(1) $R_{kjih}+R_{jikh}+R_{ikjh}=0$,

(2) $p_t p^t F_{kh}+\nabla_k q_h-2q_k p_h=0$.

Here and in the sequel we assume that the manifold M satisfies the equation (1) of lemma 1 or (3.4).

Then we have, using (1) and (2) of lemma 1,

(3.5) $$R_{kjih}=R_{ihkj},$$

(3.6) $$Q_{ji}+Q_{ij}=0.$$

From (3.1), we have

(3.7) $$K_{kjih}=R_{kjih}+g_{kh}P_{ji}-g_{jh}P_{ki}+P_{kh}g_{ji}-P_{jh}g_{ki}$$

$$+F_{kh}Q_{ji}-F_{jh}Q_{ki}+Q_{kh}F_{ji}-Q_{jh}F_{ki}+S_{kj}F_{ih}+F_{kj}T_{ih},$$

where

$$S_{kj}=-(\nabla_k q_j-\nabla_j q_k),$$

$$T_{ih}=2(p_i q_h-q_i p_h)$$

and consequently

$$S=F^{kj}S_{kj}=-2\nabla_t p^t, \tag{3.8}$$

$$T=F^{kj}T_{kj}=4p_t p^t. \tag{3.9}$$

From (3.5), using (2.1), (3.7), (3.6) and $P_{ji}-P_{ij}=0$, we find

$$(S_{kj}-T_{kj})F_{ih}-F_{kj}(S_{ih}-T_{ih})=0, \tag{3.10}$$

from which, by transvection with F^{kj},

$$(S-T)F_{ih}-n(S_{ih}-T_{ih})=0$$

or, using (3.8) and (3.9),

$$S_{ij}-T_{ij}=-(2/n)(\nabla_t p^t+2p_t p^t)F_{ij}. \tag{3.11}$$

On the other hand, from (3.6), we find

$$S_{ji}=-2Q_{ji}+p_t p^t F_{ji}. \tag{3.12}$$

From (3.11) and (3.12), using

$$P_t{}^t=\nabla_t p^t+(n/2)p_t p^t, \tag{3.13}$$

we have

$$T_{ji}=-2Q_{ji}+(2/n)(P_t{}^t+2p_t p^t)F_{ji}. \tag{3.14}$$

From (3.4) and (2.2), we find

$$\nabla_k p_j=-g_{kj}(p_t p^t)-2q_k q_j,$$

from which, using (3.13), we find

$$2P_t{}^t+(n+4)p_t p^t=0. \tag{3.15}$$

In (3.1), we contract with respect to h and k and use (3.2), (3.12), (3.14) and (3.15), then we obtain

$$K_{ji}=R_{ji}+(n+4)P_{ji}+P_t{}^t g_{ji}, \tag{3.16}$$

from which,

$$K=R+2(n+2)P_t{}^t. \tag{3.17}$$

Substituting $P_t{}^t=(K-R)/2(n+2)$ obtained from (3.17) into (3.16), we obtain

$$K_{ji}=(n+4)P_{ji}+(K-R)g_{ji}/2(n+2)+R_{ji},$$

from which,

$$P_{ji}=-L_{ji}+Rg_{ji}/2(n+2)(n+4)-R_{ji}/(n+4). \tag{3.18}$$

From (3.18), we find, using (3.2),

$$Q_{ji}=-M_{ji}-RF_{ij}/2(n+4)(n+2)+R_{jt}F_i{}^t/(n+4). \tag{3.19}$$

From (3.12), (3.15) and (3.19), we obtain

$$S_{ji}=2M_{ji}+RF_{ij}/(n+4)(n+2)-2R_{jt}F_i{}^t/(n+4)-2P_t{}^t F_{ji}/(n+4)$$

or, using (3.17),

(3.20) $S_{ji}=2M_{ji}+RF_{ij}/(n+4)(n+2)-2R_{jt}F_i^t/(n+4)-(K-R)F_{ji}/(n+4)(n+2).$

From (3.14) and (3.19), we find

$$T_{ji}=2M_{ji}+RF_{ij}/(n+4)(n+2)-2R_{jt}F_i^t/(n+4)+2P_t^tF_{ji}/(n+4)$$

or, using (3.17)

(3.21) $T_{ji}=2M_{ji}+RF_{ij}/(n+4)(n+2)-2R_{jt}F_i^t/(n+4)+(K-R)F_{ji}/(n+4)(n+2).$

Substituting (3.18), (3.19), (3.20) and (3.21) into (3.7), we find

$$\begin{aligned}&2(n+2)(n+4)B_{kjih}\\&=2(n+2)(n+4)R_{kjih}+\delta_k^h\{-2(n+2)R_{ji}+Rg_{ji}\}\\&-\delta_j^h\{-2(n+2)R_{ki}+Rg_{ki}\}+\{-2(n+2)R_k^h+R\delta_k^h\}g_{ji}-\{-2(n+2)R_j^h+R\delta_j^h\}g_{ki}\\&+F_k^h\{2(n+2)R_{jt}F_i^t-RF_{ij}\}-F_j^h\{2(n+2)R_{kt}F_i^t-RF_{ik}\}+\{2(n+2)R_{kt}F^{ht}\\&-RF^h{}_k\}F_{ji}-\{2(n+2)R_{jt}F^{ht}-RF^h{}_j\}F_{ki}-2\{2(n+2)R_{kt}F_j^t-RF_{jk}\}F_i^h\\&-2F_{kj}\{2(n+2)R_{it}F^{ht}-RF^h{}_i\}.\end{aligned}\tag{3.22}$$

Let us define $\bar{B}_{kjih}$ as following:

$$\begin{aligned}\bar{B}_{kjih}=&R_{kjih}+\delta_k^h\bar{L}_{ji}-\delta_j^h\bar{L}_{ki}+\bar{L}_k^hg_{ji}-\bar{L}_j^hg_{ki}\\&+F_k^h\bar{M}_{ji}-F_j^h\bar{M}_{ki}+\bar{M}_k^hF_{ji}-\bar{M}_j^hF_{ki}-2(\bar{M}_{kj}F_i^h+F_{kj}\bar{M}_i^h),\end{aligned}$$

where

$$\bar{L}_{ji}=-R_{ji}/(n+4)+Rg_{ji}/2(n+2)(n+4),\qquad \bar{M}_{ji}=-\bar{L}_{jt}F_i^t.$$

Thus we have the following theorem.

THEOREM 1. *If, in an n-dimensional Kaehler manifold* ($n\geqq4$), *there exists a scalar function* p *which satisfies the equation* (3.4), *then we have*

$$B_{kjih}=\bar{B}_{kjih}.\tag{3.23}$$

Suppose that there exists a scalar function p such that the complex conformal connection defined by (2.2) is of zero curvature. In this case, we can conclude that the Bochner curvature tensor of the manifold vanishes by virtue of the above theorem 1 [1].

If we assume that $B_{kjih}=0$ and $R=0$, then we have, using (3.22),

$$\begin{aligned}(n+4)R_{kjih}=&g_{kh}R_{ji}-g_{jh}R_{ki}+R_{kh}g_{ji}-R_{jh}g_{ki}-F_{kh}R_{jt}F_i^t+F_{jh}R_{kt}F_i^t\\&-R_{kt}F_h^tF_{ji}+R_{jt}F_h^tF_{ki}+2R_{kt}F_j^tF_{ih}+2F_{kj}R_{it}F_h^t.\end{aligned}\tag{3.24}$$

Conversely, if the equation (3.24) satisfies, then we have $R=0$, consequently

$$B_{kjih}=0.$$

Thus we have the following theorem.

THEOREM 2. *If, in an n-dimensional Kaehler manifold* ($n\geqq4$), *there exists a scalar function* p *such that the complex conformal connection* Γ_{ji}^h *defined by* (2.2) *satisfies the equation* (3.4) *and the curvature tensor* R_{kjih} *constructed by* Γ_{ji}^h *is of the form* (3.24), *then the Bochner curvature tensor of the manifold vanishes.*

References

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