J. Korean Math. Soc.
Vol. 11, No. 2, 1974

A BOUNDARY PROPERTY OF MEROMORPHIC FUNCTIONS WITHOUT KOEBE ARCS

BY UN HAING CHOI

Let S be a domain on the Riemann sphere Ω, and let P be a point in S, Q a point in Ω. By a suitable linear transformation L the point P can be transformed to the south pole, the point Q to the north pole. Then S is said to be (P, Q)-star-shaped if the stereographic projection of $L(S)$ is star-shaped with respect to the origin in the complex plane.

A domain S on the Riemann sphere is said to be star-shaped if it is (P, Q)-star-shaped for some point P in S and Q in Ω.

A Gross domain of a meromorphic function on D is defined to be a subdomain of D having the following properties:

(a) $f(z)$ maps G one-to-one onto a star-shaped domain S on the Riemann sphere, and

(b) G is not properly contained in any other subdomain of D having the property (a).

In [1] we proved the following

THEOREM 1. *Let $f(z)$ be a meromorphic function in the unit disc, without Koebe arcs, which has asymptotic values on a dense set in an arc α on the unit circle C. Then for each point ζ on the arc α, either,*

(a) *$f(z)$ has an asymptotic value at ζ, or*

(b) *every neighborhood of ζ contains non-degenerate Gross domains of $f(z)$, and furthermore $\delta(\zeta, d)\rightarrow 0$ as $d\rightarrow 0$, where $\delta(\zeta, d)$ denotes the supremum of the euclidean diameters of the Gross domains of $f(z)$ intersecting $\{z: |z-\zeta|<d\}$.*

The purpose of this note is to apply the above theorem to generalize a result obtained by Pommerenke and McMillan [2]. The main argument in the following proof is essentially found in [2].

We begin by showing a lemma which will be used in the proof of theorem 2.

LEMMA. *Let $f(z)$ be a meromorphic function in the unit disc, without Koebe arcs, and assume that $f(z)=h(z)/g(z)$, where $h(z)$ is an analytic function and $g(z)$ is a bounded analytic function which is not identically equal to 0. Let J_0 be an open arc on the unit circle C. Assume there exists a sequence of analytic Jordan arcs $J_n\subset D$ such that $J_n\longrightarrow J_0$ and $f(z)$ maps each J_n one-to-one onto a segment of a great circle on the Riemann sphere whose stereographic projection is a half line or a segment in the w-plane. Then $h(z)$ is bounded in a neighborhood of each point ζ of J_0, and $f(z)$ has asymptotic values at almost all points of J_0.*

Proof. By taking a suitable subarc of J_n, and by taking a subsequence of $\{J_n\}$, if necessary, we may assume that $\{J_n\}$ converges to a subarc of J_0 containing ζ and that the spherical length of each $f(J_n)$ is not greater than $\pi/2$.

Without loss of generality we may assume that the endpoints of the segments of circles

Received by the editors Oct. 30, 1974.

$f(J_n)$ converge, respectively, to the points w' and w'' on the Riemann sphere. We also assume that the "directions" of the segments of $f(J_n)$ converge, and consequently that $f(J_n)$ "converges", as n tends to ∞, to a segment L joining w' and w'' (which may be a single point if $w'=w''$).

By a suitable linear transformation we can make L the segment whose stereographic projection is on the real axis on the w-plane. Now we consider two distinct point ζ' and ζ'' on J_0, and choose points z_n' and z_n'' on J_n such that $z_n'\to\zeta'$ and $z_n''\to\zeta''$. We assume without loss of generality that the corresponding sequences of points $f(z_n')$ and $f(z_n'')$ converge. Neither of these limits is ∞, because $f(z)$ maps J_n one-to-one onto $f(J_n)$, and because $f(z)$ has no sequence of Koebe arcs for the value ∞. Therefore, by replacing J_0 by its subarc between ζ' and ζ'', we can assume that the stereographic projection of L is bounded.

We now show that $h(z)$ is bounded in a neighborhood of each point of J_0. Assume to the contrary that there exists a point ζ of J_0 and a sequence of points $z_j\in D$ such that $h(z_j)\to\infty$ as $z_j\to\zeta$. Now consider $\{h(J_n)\}$. Let L_j be the stereographic projection of the half line $\{tv(z_j):t\geq 1\}$, $v(z_j)=s(h(z_j))$ and s is the stereographic projection, and let T_j be the component of the preimage $h^{-1}(L_j)$ that contains z_j. We choose z_j so that $h'(z)\neq 0$ on T_j. Then T_j is a simple curve tending to one end to a point of $|z|=1$.

For sufficiently large j, L_j does not intersect $\cup_n h(J_n)$, because $g(z)$ is bounded and $h(z)=f(z)\cdot g(z)$, and we see that $h(z)$ has a sequence of Koebe arcs for the value ∞. Then $f(z)$ would have a sequence of Koebe arcs for the value ∞, contrary to assumption. Hence $h(z)$ is bounded in a neighborhood of ζ, thus $f(z)$ can be written as a quotient of two bounded analytic functions in a neighborhood of ζ, that is, the function $f(z)$ locally belongs to the Nevanlinna class N. The well known theorem (Privalov [4], p. 56) that a function in the class N has angular limits almost everywhere on the unit circle, completes the proof of the lemma.

Replacing lemma 1, in the proof of theorem 1, with the above lemma we obtain the following:

THEOREM 2. *Let $f(z)$ be a meromorphic function in the unit disc, without Koebe arcs, and assume that $f(z)=h(z)/g(z)$, where $h(z)$ is an analytic function and $g(z)$ is a bounded analytic function which is not identically equal to* 0. *Then for each ζ on the unit circle, either*

(a) *$f(z)$ has an asymptotic value at ζ, or*

(b) *every neighborhood of ζ contains non-degenerate Gross domains of $f(z)$, and furthermore $\delta(\zeta,d)\to 0$ as $d\to 0$.*

As corollaries we have the following known theorems.

THEOREM 3. (Pommerenke and McMillan [2]) *Let $f(z)$ be an analytic function in the unit disc, without Koebe arcs. Then for each point ζ on the unit circle, either*

(a) *$f(z)$ has an asymptotic value at ζ, or*

(b) *every neighborhood of ζ contains ternary Gross domains of $f(z)$, and furthermore $\delta(\zeta,d)\to 0$ as $d\to 0$.*

Proof. In the definition of a (P,Q)-star-shaped domain on the Riemann sphere take the north pole for Q, and let $g(z)$ be identically equal to 1 in theorem 2.

THEOREM 4. *Let $f(z)$ be a locally univalent meromorphic function without Koebe arcs. Then for each point ζ on the unit circle, either*

(a) *$f(z)$ has an asymptotic value at ζ, or*

(b) *every neighborhood of ζ contains non-degenerate Gross domains of $f(z)$, and moreover $\delta(\zeta, d)\to 0$ as $d\to 0$.*

Proof. Pommerenke and McMillan [3] have shown that a locally univalent meromorphic function without Koebe arcs has three distinct asymptotic values on each arc of the boundary of the unit disc. The theorem follows readily from this result applied to theorem 1.

References

[1] Choi, Un Haing, *On the Gross domain of a meromorphic function*, J. Korean Math. Soc. **9**(1972), 101-104.

[2] Pommerenke, Ch. and McMillan, J. E., *On the boundary behavior of analytic functions without Koebe arcs*, Math. Ann. **189**(1970), 275-279.

[3] Pommerenke, Ch. and McMillan, J. E., *On the asymptotic values of locally univalent meromorphic functions*, J. reine angew. Math., to appear.

[4] Privalov, I. I., *Randeigenschaften analytischer Funktionen*, Deutscher Verlag der Wissenschaften, Berlin, 1956.

Inha University