J. Korean Math. Soc.
Vol. 11, No. 2, 1974

SOME PROPERTIES OF PSEUDONORMABLE SEMILINEAR SPACES

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1. Introduction.

F. F. Bonsall [1], [2] and S. Bourne [3], [4] have taken a semilinear algebras to be a subset of a Banach algebra which is closed under addition, multiplication and scalar multiplication by non-negative reals. And R. E. Worth [10] has defined the topological semilinear space is a semilinear space S over R^+, the set of all non-negative reals, with a Hausdorff topology such that addition and scalar multiplication are continuous. When a space is a normed linear space over a scalar field a topology on the space is defined by its norm or by its invariant metric. However this is not the case for a semilinear space, for the pseudonorm does not define an invariant pseudo-metric on the space. But we confine our study to a semilinear space with a pseudo-norm.

In this paper, we, with a topological structure of a semilinear space whose topology is not assumed to be Hausdorff, study a part of the classical theory of pseudonormable semilinear spaces. §2 is a summary of relavent concepts and theorems which are employed in a part of the remainder of the paper. In §3 the general theory is developed. In §4 the quotient semilinear spaces are developed.

2. Preliminaries.

In this paper we shall use the terms in the sense given by the author in previous paper [13]. For the sake of completeness we repeat:

DEFINITION 1. A Half-field is a system consisting of a set H and two binary operations called addition and multiplication with the following properties:

(1) H is a semi-group with identity (0) under addition.

(2) H with non-zero elements forms a commutative group under the multiplication.

(3) The left-hand and right-hand distributive laws:

$$a(b+c)=ab+ac \quad \text{and} \quad (a+b)c=ac+bc$$

hold for every elements a, b, and c of H.

DEFINITION 2. A topological half-field is a half-field H together with a topology on H under which the half-field operations are continuous.

DEFINITION 3. A semilinear space over a scalar half-field R^+ is a system of a set E and two operations of addition and scalar multiplication by R^+ with the following properties:

(1) E is a commutative and cancellative semigroup under the addition with the additive identity zero (0).

(2) For every x and y of E and for every α of R^+, $\alpha(x+y)=\alpha x+\alpha y$.

(3) For every x of E and for every α and β of R^+,

Received by the editors Oct. 30, 1974

$$(\alpha+\beta)x=\alpha x+\beta x \quad \text{and} \quad \alpha(\beta x)=(\alpha\beta)x.$$

(4) For every x of E and for every α of R^+, $\alpha x=x\alpha$.

(5) For every x of E $1x=x$ and $0x=0$ where the zero of left-hand is the zero of scalar and that of right-hand is the zero of E.

DEFINITION 4. A subset of a semilinear space over scalar half-field R^+ is said to be a semilinear subspace of the space if the addition and scalar multiplication operations are closed in the set.

DEFINITION 5. A subset A of a semilinear space E over a scalar half-field R^+ is said to be convex if for every pair of x and y of A and for every pair of α and β of R^+ with $\alpha+\beta=1$, $\alpha x+\beta y\in A$.

DEFINITION 6. A pseudo-metric for a semilinear space E over a scalar half-field R^+ is a non-negative real valued function d such that:

(1) $d(x,x)=0$ for every x of E,

(2) For every x, y, and z of E, $d(x,y)\leq d(x,z)+d(y,z)$.

When a pseudo-metric d satisfies that $d(x,y)=0$ implies $x=y$ we say that d is a metric for E.

Here we remark that:

(1)For every x and y of E, $d(x,y)=d(y,x)$.

(2) Every metric is a pseudo-metric.

DEFINITION 7. A pseudo-metric (metric) d for a semilinear space over a scalar half-field R^+ is said to be pseudo-metric invariant (metric invariant resp.) if for every x, y and z of E and for every α of R^+, $d(\alpha x+y,\ \alpha z+y)=\alpha d(x,z)$.

DEFINITION 8. A pseudo-norm on a semilinear space E over a scalar half-field R^+ is a non-negative real valued function ν, defined for each point of E, such that:

(1) For each pair of points of E, $\nu(x+y)\leq\nu(x)+\nu(y)$.

(2) For every α of R^+ and for every x of E, $\nu(\alpha x)=\alpha\nu(x)$.

When a pseudo-norm ν satisfies that $\nu(x)=0$ implies $x=0$ we say that ν is a norm.

THEOREM 1. *In a semilinear space E over a scalar half-field R^+ an invariant pseudo-metric (invariant metric) defines a pseudo-norm (norm resp.).*

Proof. Let d be an invariant metric and we define $\nu(x)=d(x,0)$ for every x of E. Then we have

$$\nu(x+y)=d(x+y,0)\leq d(x+y,y)+d(y,0)=d(x,0)+d(y,0)=\nu(x)+\nu(y),$$

and

$$\nu(\alpha x)=d(\alpha x,0)=\alpha d(x,0)=\alpha\nu(x)$$

for every α of R^+ and for every x and y of E. If d is an invariant metric, then $\nu(x)=d(x,0)=0$ implies $x=0$. This completes the proof.

DEFINITION 9. A topological semilinear space is a semilinear space E over a scalar half-field R^+ together with a topology in E such that: addition : $E\times E\longrightarrow E$ and scalar multiplication : $R^+\times E\longrightarrow E$ are continuous with respect to the topology in E

and the product topologies in $E\times E$ and in $R^+\times E$.

Here we remark that:

(1) The continuity of addition is that for any a and b of E and for any neighborhood U of $a+b$ there exist a neighborhood V of a and a neighborhood W of b such that $V+W\subseteq U$, and that of multiplication is that for any a of E and for any α of R^+ and for any neighborhood U of αa there exist a neighborhood A of α and a neighborhood V of a such that $AV\subseteq U$.

(2) Any neighborhood of x of E is denoted by $x+V$ where V is a neighborhood of the origin 0.

DEFINITION 10. A topological semilinear space is locally convex if for every neighborhood U of the origin there exists a convex neighborhood V of the origin such that $V\subseteq U$.

DEFINITION 11. A subset B of a topological semilinear space is bounded if every neighborhood U of the origin there exists a positive integer n such that $B\subseteq nU$.

DEFINITION 12. A topological semilinear space is locally bounded if there exists a bounded neighborhood of the origin.

THEOREM 2. *Every locally bounded topological semilinear space E is a first countable space.*

Proof. It is sufficient to prove that it has a countable base at the origin. By definition E has a bounded neighborhood U of the origin. Let $\mu_\alpha : E\longrightarrow E$ be a function such that $\mu_\alpha(x)=\alpha x$ for every x of E and for every α of positive real numbers. Since the scalar multiplication is continuous, so is μ_α. By the identity $\mu_\alpha\circ\mu_{\alpha^{-1}}=\mu_{\alpha^{-1}}\circ\mu_\alpha=I_E$, we have μ_α is a homeomorphism. Furthermore $\mu_\alpha(U)=\{\alpha x\,|\,x\in U\}=\alpha U$ is a neighborhood of the origin for every α of the positive real numbers. Let V be any neighborhood of the origin. Then there exists a positive integer n such that $U\subseteq nV$, that is, $\frac{1}{n}U\subseteq V$, hence $\{\frac{1}{n}U\,|\,n\in N\}$ forms a countable base at the origin. This completes the proof.

3. Pseudonormable semilinear spaces.

In a linear space E over R^+ if ν and d are pseudo-norm and an invariant pseudo-metric for E respectively, then they are definable each other by $d(\alpha x, \beta y)=\nu(\alpha x-\beta y)$ for every x and y of E and for every α, β of R^+.

THEOREM 3. *Every semi-linear space E over a scalar half-field R^+ with an invariant pseudo-metric is a topological semilinear space with the invariant pseudo-metric topology, more precisely a locally convex and locally bounded (topological) semilinear space.*

Proof. By Theorem 1 since

$$d(x+y, a+b)\leqq d(x+y, y+a)+d(y+a, a+b)=d(x, a)+d(y, b),$$

the addition is continuous. Let β be any element of a neighborhood of a scalar α and y any element of neighborhood of an x of the space, then, since $\alpha>\beta$ implies that there exists an $\varepsilon>0$ such that $\alpha=\beta+\varepsilon$, we have

$$d(\alpha x, \beta y)\leq d(\alpha x, \alpha y)+d(\alpha y, \beta y)$$
$$=\alpha d(x, y)+d(\beta y+\varepsilon y, \beta y) =\alpha d(x, y)+\varepsilon d(y, 0),$$

which proves that the scalar multiplication is continuous. Let $B_\rho=\{x|d(x, 0)<\rho\}$ be any neighborhood of the origin and $y, z\in B_\rho$ and $\alpha+\beta=1$. Then we have, by setting $\nu(x)=d(x, 0)$, $\nu(\alpha y+\beta z)\leq\alpha\nu(y)+\beta\nu(z)<\alpha\rho+\beta\rho=\rho$, that is, $\alpha y+\beta z\in B_\rho$, which shows the local convexity of the space. Let U be any neighborhood of the origin. Then there exists a $B_\rho=\{x|d(x, 0)<\rho\}\subseteq U$. Hence $B_1=\frac{1}{\rho}B_\rho\subseteq\frac{1}{\rho}U$, and since there exists a positive integer n such that $\frac{1}{\rho}<n$, we have $B_1\subset nU$, which shows the local boundedness of the space. This completes the proof.

DEFINITION 13. A topological semilinear space defined by a pseudo-norm (norm) is called a pseudonormed (normed) semilinear space.

We remark that ν, defined by $\nu(x)=d(x, 0)$, is a pseudonorm (norm resp.) for the space.

DEFINITION 14. A topological semilinear space is pseudonormable (normable) if and only if there exists a pseudo-norm (norm resp.) whose topology is that of the space.

We remark that the topology of a normable semilinear space is Hausdorff, and that any normable topological semilinear space is pseudonormable.

LEMMA 1. *In a topological semilinear space E with the topology $\mathcal{T}$ if a subset B of E is convex then so is the interior $Int(B)$ of B.*

Proof. Since $Int(B)=\bigcup\{U\in\mathcal{T}|U\subset B\}$, for any x and y of $Int(B)$ there exist a neighborhood U of x and a neighborhood V of y such that $U\subseteq B$ and $V\subseteq B$. Since for α and β with $\alpha+\beta=1$ in R^+,

$$\alpha U+\beta V\subseteq\alpha B+\beta B\subseteq B,\ \ \alpha x+\beta y\in Int(B).$$

THEOREM 4. *A topological semilinear space E is pseudonormable if and only if it is locally convex and locally bounded.*

Proof. "Only if" is obvious by Theorem 3. "If": Assume that the condition holds. By the remark of Definition 9 and by Theorem 3 it is sufficient to consider at the origin. By Lemma 1 there exists a bounded convex neighborhood V of the origin, and clearly there exists an $\alpha\in R^+$ such that $x\in\alpha V$ for every x of E. Let σ be a relation such that $x\sigma y$ if and only if there exist an $\alpha\in R^+$ and $v\in\alpha V$ such that $x+v=y+v$. It is easy to check that σ is an equivalence relation. We denote $\nu(x)=\nu(y)$ for each pair of x and y of E which satisfies $x\sigma y$. Let's define $\nu(x)=inf\{\rho\in R^+|x\in\rho V\}$.

Then for any x and y of E and for arbitrary $\varepsilon>0$,

$$x\in(\nu(x)+\varepsilon)V \ \text{ and } \ y\in(\nu(y)+\varepsilon)V.$$

Since V is convex,

$$\frac{\alpha x}{\nu(x)+\varepsilon}+\frac{\beta y}{\nu(y)+\varepsilon}\in V \text{ for } \alpha+\beta=1 \text{ in } R^+.$$

By setting $\alpha=\frac{\nu(x)+\varepsilon}{\nu(x)+\nu(y)+2\varepsilon}$ we have $\beta=\frac{\nu(y)+\varepsilon}{\nu(x)+\nu(y)+2\varepsilon}$ hence $x+y\in(\nu(x)+\nu(y)+2\varepsilon)V$, that is, $\nu(x+y)\leqq\nu(x)+\nu(y)$.
By definition of ν we have, for any α of R^+, $\nu(\alpha x)=\alpha\nu(x)$. This shows that ν is a pseudonorm for E.
Let $\mathscr{T}$ be the given topology and ξ a topology for E defined by the pseudo-norm ν. We assert that $\xi=\mathscr{T}$.
Let $G\in\mathscr{T}$ be any neighborhood of the origin. By the boundedness of V, there exists a positive integer n such that $\frac{1}{n}V\subseteq G$ and $\{x\mid\nu(x)<\frac{1}{n}\}\subseteq\frac{1}{n}V\subseteq G$, hence $G\in\xi$, that is, $\mathscr{T}\subseteq\xi$. Let H be a neighborhood of the origin with respect to ξ. Then there exists a $\rho\in R^+$ such that $\{x\mid\nu(x)<\rho\}\subseteq H$. Since there exists a $\delta\in R^+$ such that $\nu(x)<\delta<\rho$, we have $\delta V\subseteq\{x\mid\nu(x)<\rho\}\subseteq H$, and δV being open with respect to $\mathscr{T}$, $H\in\mathscr{T}$, that is, $\xi\subseteq\mathscr{T}$. This completes the proof.

By Theorem 2 and Theorem 4 we have the following

COROLLARY. *Every pseudonormable semilinear space is first countable.*

We remark that, since the property of a space being first countable is hereditary, any subspace of the pseudonormable semilinear space is first countable.

DEFINITION 15. A non-empty subset B of a directed set D with a binary relation σ which directs D is residual if there exists an element d of D such that $d\sigma a$ implies $a\in B$.

We note that $T_d=\{a\in D\mid d\sigma a\}$ is residual.

DEFINITION 16. A net in a topological semilinear space E is a function $\varphi: D\longrightarrow E$ of some directed set D, and we say that φ converges to a (denoted by $\varphi\longrightarrow a$) if for every neighborhood U of a there exists a residual set T_d such that $\varphi(T_d)\subseteq U$, and a net $\varphi: N\longrightarrow E$ is called a sequence in E.

DEFINITION 17. A closed subspace of a pseudonormed (normed) semilinear space E is a topological semilinear subspace of the semilinear space E which is a closed subset of the pseudonormed (normed resp.) semilinear space E.

THEOREM 5. *The closure $cl(X)$ of a semilinear subspace X of a pseudonormable (normable) semilinear space E is a convex subspace of E.*

Proof. Since E is first countable, for every x of $cl(X)$ there exists a decreasing countable base $\{U_j\mid j\in N\}$ containing x. Since $x\in cl(X)$ if and only if there exists a net φ in X which converges to x (See, [7]), $x\in cl(X)$ if and only if there exists a directed set D such that $\varphi: D\longrightarrow X$ which converges to x, that is, for every U_j there exists a residual set T_a such that $\varphi(T_a)\subseteq U_j$. Choose $x_j\in\varphi(T_a)\subseteq U_j$ $(j\in N)$. Hence, by setting $\varphi(j)=x_j$ φ is a sequence in X with $\varphi\longrightarrow x$, which shows that $x\in cl\ (X)$ if and only if there exists a sequence φ in X converging to x. Let $a\in cl(X)$ and $b\in cl(X)$. Then there exist sequences f and g in X which converges to a and b respectively, hence sequences $f+g$ and $\alpha f(\alpha\in R^+)$ in X converges to $a+b\in cl(X)$ and $\alpha a\in cl(X)$ respectively. This completes the proof.

Since any pseudonormed (normed) semilinear space is pseudonormable (normable resp.), we have the following

COROLLARY. *The closure $cl(X)$ of a pseudonormed (normed) semilinear subspace X of the space E is a convex subspace.*

THEOREM 6. *If a pseudonormable semilinear space E is a T_1 space then E is a normable semilinear space, and vice versa.*

Proof. Let ν be a defining pseudonorm for E, and $x\neq 0$ in E. Since E is T_1, $\{x\}$ is closed. By setting $E\setminus\{x\}=G$, G is open and contains 0. Hence there exists a locally convex and locally bounded neighborhood V of 0 such that $x\notin V$. Hence $\nu(x)\neq 0$. This shows that ν is a norm for E, thus E is normable. Conversely, if ν is a defining norm for E, then $x\neq 0$ in E implies that $\nu(x)\neq 0$. Hence for $H=E\setminus\{x\}$ there exists a neighborhood V of 0 such that $y+V\subseteq H$ for any y in H, thus $\{x\}$ is closed. This completes the proof.

Since the normability carries the Hausdorff topology we have the following

COROLLARY. *A pseudonormable semilinear space is a T_1 space if and only if it is a Hausdorff semilinear space.*

4. Quotient semilinear spaces.

Let E be a semilinear spaces over R^+ and S a semilinear subspece of E and σ the relation defined by the statement: $a\sigma b$ if and only if there exist s_1 and s_2 of S such that $a+s_1=b+s_2$ which is denoted by $a\equiv b$ mod S. It is easy to check that σ is an equivalence relation in E. We denote by E/S the quotient set of E modulo the equivalence relation σ defined by S.

DEFINITION 18. Let $(X,\mathscr{T})$ be a topological space and Y any set and p; $X\longrightarrow Y$ a function. The identification topology in Y determined by p is $\mathscr{T}_p=\{G\subset Y\,|\,p^{-1}(G)\in\mathscr{T}\}$.

We remark that $\mathscr{T}_p$ is the largest topology in Y for which $p: X\longrightarrow Y$ is continuous.

THEOREM 7. *Let E be a pseudonormable semilinear space and S a semilinear subspace of E, $p: E\longrightarrow E/S$ a canonical projection where E/S has an identification topology. Then p is a continuous linear function and E/S is pseudonormable.*

The space E/S is called the quotient pseudonormable semilinear space of E (modulo the equivalence relation defined) by a subspace S

Proof. Since E/S has an identification topology, the continuity of p is clear (See, for example, [5]). Linearity of p: Let $\alpha\in R^+$ and $x,y\in E$. Then

$$p(x)+p(y)=\{a\,|\,a\in p^{-1}p(x)\}+\{b\,|\,b\in p^{-1}p(y)\}=\{a+b\,|\,a\in p^{-1}p(x),\ b\in p^{-1}p(y)\}$$
$$=\{a+b\,|\,a\equiv x \text{ and } b\equiv y,\ mod\ S\}=\{a+b\,|\,a+b\in p^{-1}p(x+y)\}=p(x+y),$$
$$\alpha p(x)=\{\alpha a\,|\,a\in p^{-1}p(x)\}=\{\alpha a\,|\,a\equiv x\ mod\ S\}=\{\alpha a\,|\,\alpha a\equiv\alpha x\ mod\ S\}$$
$$=\{\alpha a\,|\,\alpha a\in p^{-1}p(\alpha x)\}=p(\alpha x),$$

which shows p is linear. With the linearity of p it is easy to see the semilinearity of

E/S.

Construction of a pseudo-norm which defines the identification topology: Let's define ν^* as follows: $\nu^*(p(x))=inf\{\nu(a)\,|\,a\in p^{-1}p(x)\}$ where ν is the pseudonorm for E which defines the topology of the space to be pseudonormed. It is clear that ν^* is well-defined and $\nu^*(p(x))\geqq 0$ for every $p(x)$ of E/S. Let $\varepsilon>0$ be arbitrary, then there exist an $a\in p^{-1}p(x)$ and a $b\in p^{-1}p(y)$ such that $\nu^*(p(x))+\frac{1}{2}\varepsilon>\nu(a)$ and $\nu^*(p(y))+\frac{1}{2}\varepsilon>\nu(b)$, and hence $\nu^*(p(x))+\nu^*(p(y))+\varepsilon>\nu(a)+\nu(b)\geqq\nu(a+b)\geqq inf\{\nu(a+b)\,|\,a\in p^{-1}p(x), b\in p^{-1}p(y)\}=\nu^*(p(x)+p(y))$, since $\varepsilon>0$ is arbitrary we have $\nu^*(p(x)+p(y))\leq\nu^*(p(x))+\nu^*(p(y))$, and for $\alpha\in R^+$, $\nu^*(\alpha p(x))=\alpha\nu^*(p(x))$ is obvious by the definition of ν^*. Thus ν^* is a pseudonorm for E/S. We now prove that the topology defined by $\nu*$ is the identification topology in E/S. Let U be any neighborhood of the origin in E/S with respect to the identification topology. Then $p^{-1}(U)$ is a neighborhood of the origin in E. Hence there exists a convex and bounded neighborhood V of the origin in E such that $V\subseteq p^{-1}(U)$, and there exists a $\rho\in R^+$ such that $V=\{x\,|\,\nu(x)<\rho\}$. Then since $x\in p^{-1}(U)$ implies that $p(x)\in U$, $\nu(x)<\rho$ implies that $\{p(a)\,|\,a\in V\}\subseteq U$. This shows that U is also contained in the topology defined by ν^*. This completes the proof.

By Theorem 6 we have the following

COROLLARY. *The quotient pseudonormable semilinear space E/S of a pseudonormable semilinear space E modulo a semilinear subspace S is Hausdorff if and only if S is closed.*

Proof. Necessity: Since E/S is Hausdorff $\{0\}$ in E/S is closed. By the continuity of the canonical projection $p: E\longrightarrow E/S$ $p^{-1}(0)=S$ is closed in E.
Sufficiency: Straightforward from Theorem 6.

THEOREM 8. *Let E_j $(j=1,2)$ be pseudonormable semilinear spaces, and as binary operations in the product $E_1\times E_2$*

$$\text{addition: } (a,b)+(x,y)=(a+x,\ b+y),$$
$$\text{scalar multiplication: } \alpha(a,b)=(\alpha a,\alpha b)$$

are given, and let $d_j(j=1,2)$ be pseudo-norms which define the topologies of E_j respectively. Then the function

$$d_1\times d_2;\ (a,b)\longrightarrow d_1(a)+d_2(b)$$

is a pseudo-norm for the product space $E_1\times E_2$ which defines the topology of $E_1\times E_2$, hence $E_1\times E_2$ is a pseudonormable semilinear space.

Proof. The semilinearity of the product $E_1\times E_2$ is the consequence of the binary operations given. By definition of $d_1\times d_2$,

1. $d_1\times d_2(a,b)=d_1(a)+d_2(b)\geq 0$ for every element (a,b) of $E_1\times E_2$.
2. $d_1\times d_2[(a,b)+(x,y)]=d_1\times d_2(a+x,\ b+y)=d_1(a+x)+d_2(b+y)$
$$\leqq d_1(a)+d_1(x)+d_2(b)+d_2(y)=d_1\times d_2(a,b)+d_1\times d_2(x,y)$$
for every pair of (a,b) and (x,y) of $E_1\times E_2$.
3. $d_1\times d_2(\alpha(a,b))=d_1\times d_2(\alpha a,\alpha b)=d_1(\alpha a)+d_2(\alpha b)$
$$=\alpha d_1(a)+\alpha d_2(b)=\alpha d_1\times d_2(a,b)$$

for every α of R^+ and for every (a, b) of $E_1 \times E_2$.

This shows that $d_1 \times d_2$ is a pseudonorm for $E_1 \times E_2$. Let $U_1 \times U_2$ be any neighborhood of $(a_1, a_2) + (b_1, b_2)$.

Then $U_j (j=1, 2)$ are neighborhoods of $a_j + b_j$ $(j=1, 2)$ respcetively, so that there exist neighborhoods V_j of a_j and neighborhoods W_j of b_j such that $V_j + W_j \subseteq U_j$ $(j=1, 2)$ respectively. Hence we have $U_1 \times U_2 \supseteq (V_1 + W_1) \times (V_2 + W_2) \supseteq V_1 \times V_2 + W_1 \times W_2$.

Since $V_1 \times V_2$ and $W_1 \times W_2$ are neighborhoods of (a_1, a_2) and (b_1, b_2) respectively, the continuity of addition is proved.

The continuity of scalar multiplication can be proved in a similar way.

Let $V \times W$ be any neighborhood of (x_1, x_2) with respect to the product topology. Then $P_{r,j}(G_1 \times G_2) = G_j$ (where $P_{r,j}$ is the projection onto j-th factor) implies that there exists a convex and bounded neighborhood H_j of x_j with respect to the topology defined by d_j such that $H_j \subseteq G_j$ $(j=1, 2)$ respectively. This completes the proof.

Let E be a pseudonormable semilinear space and $S = \{(x, x) \mid x \in E\}$ and σ the relation defined by the statement: $(a, b)\sigma(c, d)$ if and only if there exist (x, x) and (y, y) of S such that $(a, b) + (x, x) = (c, d) + (y, y)$. It is easy to check that σ is an equivalence relation, that is, $(a, b) \equiv (c, d)$ mod S.

By Theorem 7 we have the following

COROLLARY. *Let E be a pseudonormable semilinear space and $S = \{(x, x) \mid x \in E\}$. If $p : E \times E \longrightarrow E \times E/S$ is a canonical projection where $E \times E/S$ has an identification topology. Then $E \times E/S$ is the quotient pseudonormable semilinear space.*

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