J. Korean Math. Soc.
Vol. 11, No. 2, 1974

ON THE ATOMS OF GROUP VALUED MEASURES

BY JAIHAN YOON

1. The problems treated in the present paper are a group valued measure version of results, in particular, on atoms of real measures developed in [3]. The group valued measures in general lack some properties that play an important role in deriving such results of real measures that we are interested in here. Therefore, we have imposed a restriction upon the group valued measures that resembles the monotonity of real measures. With this restriction, one can define a uniformity on the ring of sets on which the group valued measure is defined. And it is shown that the associated uniform space is sequentially complete provided the range group is metric. Also it is shown that there exists a relation between the nonexistence of atoms and the connectedness of the associated space. This, as it happens, resembles the results obtained by Landers [1] in part.

2. In what follows, G denotes a Hausdorff group and $\mathscr{U}(e)$ the neighborhood filter of the identity e of the group G. Let $\mathbf{R}$ be a ring of subsets of a set S. A set function μ: $\mathbf{R} \rightarrow G$ is a *group valued measure* or simply a *measure* unless otherwise stated, if $\mu(\bigcup_{n=1}^{\infty} E_n) = \lim \mu(E_1) \cdot \mu(E_2) \cdots \mu(E_n)$ for every disjoint sequence $\{E_n\}$ in $\mathbf{R}$ such that $\bigcup_{n=1}^{\infty} E_n \in \mathbf{R}$. A set E with $\mu(E) \neq e$ is an atom if $\mu(F) = e$ or $\mu(F) = \mu(E)$ whenever $F \subset E$ and F in $\mathbf{R}$.

Recall that a sequence of sets is *monotone* if it is increasing or decreasing. The same arguments employed in [3, Theorem D, Theorem E, p. 39] prove the following Theorems.

THEOREM. *If* $\{E_n\}$ *is a monotone sequence in* $\mathbf{R}$ *such that* $\lim E_n \in \mathbf{R}$, *then* $\lim \mu(E_n) = \mu(\lim E_n)$.

THEOREM. *A set function* μ: $\mathbf{R} \rightarrow G$ *which is finitely multiplicative is a measure if and only if it is continuous from below at every* E *in* $\mathbf{R}$, *or continuous from above at the empty set* ϕ.

3. Motivated by Lemma 1 of [2], we shall impose a restriction upon group valued measures so that some of results concerned with atoms of real measures remain true.

DEFINITION. *A measure* μ: $\mathbf{R} \rightarrow G$ *is said to be monotone if* $\mu(E) \in U$ *for* $E \in \mathbf{R}$ *and* $U \in \mathscr{U}(e)$, *then* $\mu(F) \in U$ *whenever* $F \subset E$ *and* $F \in \mathbf{R}$.

It is straightforward from the above definition that $\mu(F) = e$ if F is a measurable subset of a measurable set E with $\mu(E) = e$, and that a pair of disjoint measurable sets cannot assume non-identity values one of which is the inverse of the other.

In what follows by a measure μ: $\mathbf{R} \rightarrow G$ we shall mean a monotone measure. In order to define a uniformity on $\mathbf{R}$, let, for each U in $\mathscr{U}(e)$, $\tilde{U}$ be the set of those pairs $(E, F) \in \mathbf{R} \times \mathbf{R}$ with $\mu(E \triangle F) \in U$, and let $\tilde{\mathscr{U}}$ be the collection of all such $\tilde{U}$. To begin

Received by the editors July. 1, 1974

with, it will be convenient to prove the following lemma.

LEMMA 1. *Let W be a neighborhood of e with $W^2 \subset U$ where $U \in \mathcal{U}(e)$. If $\mu(E\triangle X) \in W$ and $\mu(X\triangle F) \in W$, then $\mu(E\triangle F) \in U$ whenever E, F and X are in* $\mathbf{R}$.

Proof. Clearly, we have

$$
\begin{aligned}
E\triangle F &= (E-F) \cup (F-E) \\
&= ([E-(E\cap F)]\cap X) \cup ([F-(E\cap F)]-X) \cup \\
&\quad ([F-(E\cap F)]\cap X)] \cup ([E-(E\cap F)]-X),
\end{aligned}
$$

If Z stands for E or F, it is clear that

$$((Z-(E\cap F))\cap X)\cap((E\cap F)-X)=\phi,$$

$$((Z-(E\cap F))-X)\cap(X-(E\cap F))=\phi,$$

and hence

$$(Z-(E\cap F))\cap X=(((Z-(E\cap F))\cap X)\cup((E\cap F)-X))-((E\cap F)-X),$$

$$(Z-(E\cap F))-X=(((Z-(E\cap F))-X)\cup(X-(E\cap F)))-(X-(E\cap F)).$$

Replacing Z by E and F in the last two equalities and substituting sets thus obtained in the first equality, we have

$$E\triangle F=((E\triangle X)-T)\cup((F\triangle X)-T),$$

where $T=(X-(E\cap F))\cup((E\cap F)-X)$. By taking account of the monotonity of μ, we have $\mu(E\triangle F) \in U$. This completes the proof.

THEOREM 2. *Let μ: $\mathbf{R}\to G$ be a measure, then $\tilde{\mathcal{U}}$ is a base of a uniformity for $\mathbf{R}$ and μ is uniformly continuous with respect to the topology induced from $\tilde{\mathcal{U}}$.*

Proof. Averting to Lemma 1, it is obvious that $\tilde{\mathcal{U}}$ is a base of a uniformity. To show that μ is uniformly continuous, let U be any neighborhood of e. Choose a symmetric neighborhood V of e such that $V^2\subset U$. If $(E, F) \in V$, that is $\mu(E\Delta F) \in V$, then $\mu(E-F) \in V$ and $\mu(F-E) \in V$. Hence, we have

$$
\begin{aligned}
\mu(E)^{-1}\cdot\mu(F) &= \mu((E-F)\cup(E\cap F))^{-1}\cdot\mu((F-E)\cup(E\cap F)) \\
&= \mu(E-F)^{-1}\cdot\mu(F-E).
\end{aligned}
$$

Thus $\mu(E)^{-1}\cdot\mu(F) \in U$, which completes the proof.

LEMMA 3. *If $E \in \mathbf{R}$ is a given set, the mapping f_E: $X\to E\cap X$ for each $X \in \mathbf{R}$ is uniformly continuous on* $\mathbf{R}$.

Proof. This is a straightforward consequence of the equality $(X\cap E)\triangle(Y\cap E)=E\cap(X\triangle Y)$.

THEOREM 4. *Let μ:$\mathbf{R}\to G$ be a measure. If the associated uniform space in accordance with Theorem 2 is connected, then μ has no atom.*

Proof. Let A be an atom, by Theorem 2 and Lemma 3, it follows that the inverse image $(\mu\circ f_A)^{-1}(e)$ is a closed set. We shall show that it is also open. To do this let

U be any neighborhood of $e \in G$ such that $\mu(A) \in U$. If $X_0 \in (\mu \circ f_A)^{-1}(e)$, we have for each $X \in \tilde{U}(X_0)$, $\mu \circ f_A(X) = \mu(A \cap X) = \mu((A \cap X) - X_0) \cup (A \cap X \cap X_0)) = \mu((A \cap X) - X_0)$. Since $(A \cap X) - X_0 \subset X \triangle X_0$, we can not have $\mu(A \cap X) = \mu(A)$. Thus $\mu \circ f_A(X) = \mu(A \cap X) = e$, which proves $\tilde{U}(X_0) \subset (\mu \circ f_A)^{-1}(e)$.

Since $\mathbf{R}$ is connected we must have $\mathbf{R} = (\mu \circ f_A)^{-1}(e)$. This is impossible because $A \bar{\in} (\mu \circ f_A)^{-1}(e)$, completing the proof.

Henceforth, $\mathbf{S}$ will stand for a σ-ring of subsets of a given set.

THEOREM 5. *If the neighborhood filter $\mathscr{U}(e)$ has a countable base, then the associated uniform space* $\mathbf{S}$ *is sequentially complete.*

Proof. Let $\{U_n\}$ be a countable base for $\mathscr{U}(e)$. It is sufficient to show that there exists a convergent subsequence of any given Cauchy sequence $\{E_n\}$. Without loss of generality, we may assume that $U_n \supset U_{n+1}$ for all n. For each n, let $\{U_{n,i}\}$ be a sequence of neighborhoods of e such that $U^2_{n,i+1} \subset U_{n,i}$, $U_{n,1}{}^2 \subset U_n$, where $i=1, 2, \cdots$. Clearly, there exists a subsequence $\{E_{1,i}\}$ of $\{E_n\}$ such that $\mu(F_{1,\lambda} \triangle E_{1,\nu}) \in U_{1,m}$ if $m \leqslant \lambda, \nu$. This is the first step in the construction of the following subsequences represented by an array

$$S_1: \; E_{1,1}, \;\; E_{1,2}, \cdots$$

$$S_2: \; E_{2,1}, \;\; E_{2,2}, \cdots$$

$$\cdots$$

such that

(a) S_k is a subsequence of S_{k-1}

(b) $\mu(E_{k,\lambda} \triangle E_{k,\nu}) \in \bigcap_{i=1}^{k} U_{i,p}$ if $\lambda, \nu \geqslant p$.

Since S_n is a Cauchy sequence, there is an integer j_1 such that $\mu(E_{n,\lambda} \varDelta E_{n,\nu}) \in \bigcap_{i=1}^{n+1} U_{i,1}$ if $\lambda, \nu \geqslant j_1$. Write $E_{n+1,1} = E_{n,j_1}$. By an induction, we can find a sequence

$$S_{n+1}: \; E_{n+1,1}, \;\; E_{n+1,2}, \cdots$$

of S_n such that $E_{n+1,p} = E_{n,j_p}$ where j_p is an integer satisfying $\mu(E_{n,\lambda} \varDelta E_{n,\nu}) \in \bigcap_{i=1}^{n+1} U_{i,p}$ for $\lambda, \nu \geqslant j_p$ and $j_p > j_{p-1}$ $(p>1)$.

It is clear that S_{n+1} satisfies (b). In fact, for each p,

$$\mu(E_{n+1,\lambda} \triangle E_{n+1,\nu}) = \mu(E_{n,j_\lambda} \triangle E_{n,j_\nu}) \in \bigcap_{i=1}^{n+1} U_{i,p}$$

whenever $\lambda, \nu \geqslant p$.

We now go down the diagonal of the array: i.e., we consider the sequence

$$S: \; E_{1,1}, \;\; E_{2,2}, \ldots$$

Let $E = \lim\sup E_{n,n}$ and let $B_n = \bigcup_{\lambda=n}^{\infty} E_{\lambda,\lambda}$. Then, $E = \lim B_n$ and

$$\begin{aligned} E_{n,n} \triangle E &= (E_{n,n} \triangle B) \triangle (B_n \triangle E) \\ &= (E_{n,n} \triangle (\textstyle\bigcup_{\lambda=n}^{\infty} E_{\lambda,\lambda})) \triangle (B_n \triangle E) \\ &\subset (\textstyle\bigcap_{k=1}^{\infty} (E_{n+k,n+k} \triangle E_{n+k-1,n+k-1})) \triangle (B_n \triangle E). \end{aligned}$$

Because $n+k \leqslant j_{n+k}$ and $E_{n+k,n+k} = E_{n+k-1,j_{n+k}}$, clearly we have

$$\mu(E_{n+k,\, n+k} \triangle E_{n+k-1,\, n+k-1}) = \mu(E_{n+k-1,\, n+k-1} \triangle E_{n+k-1,\, j_{n+k}})$$
$$\in \bigcap_{i=1}^{n+k-1} U_{i,\, n+k-1} \subset U_{n,\, n+k-1}$$

for $k \geqslant 1$. For the simplicity of notation, write, for each $n \geqslant 1$,

$$K = \bigcup_{k=1}^{\infty} (E_{n+k,\, n+k} \triangle E_{n+k-1,\, n+k-1}), \quad K_m = \bigcup_{k=1}^{m} (E_{n+k,\, n+k} \triangle E_{n+k-1,\, n+k-1}).$$

Thus, for every $m \geqslant 1$, $\mu(K_m) \in U_{n,n} \cdot U_{n,n+1} \cdots U_{n,m} \subset U_{n,n-1}$ and hence $\mu(K_m) \in U_{n,1}$. Since $\mu(K) = \lim \mu(K_m)$, we have $\mu(K) \in \bar{U}_{n,1} \subset U_n$ and therefore $\mu(E_{n,n} \triangle (\bigcap_{\lambda=n}^{\infty} E_{\lambda,\lambda}) \in U_n$. Now let V be a given neighborhood of e and let W be a neighborhood of e such that $W^2 \subset V$. Since $\{B_n \triangle E\}$ is a decreasing sequence and converges to the empty set, there exists an integer N such that $\mu(B_n \triangle E) \subset W$ for all $n \geqslant N$. Moreover we may assume that $U_N \subset W$, which implies $\mu(E_{n,n} \triangle E) \in V$ for all $n \geqslant N$ completing the proof.

Let a measure μ: $\mathbf{S} \to G$ satisfy the following condition

(*) for each $E \in \mathbf{S}$ and $U \in \mathcal{U}(e)$ there exists a sequence $\{E_i\}$ (depending on U) such that $E = \bigcup_{i=1}^{\infty} E_i$ and $\mu(E_i) \in U$.

Since the measure μ is monotone, the above sequence $\{E_i\}$ may well be assumed disjoint and constant from some term on: i. e., every $E \in \mathbf{S}$ can be represented as $E = \bigcup_{i=1}^{n} E_i$ and $\mu(E_i) \in U$ (n, the number of E_i's, depends on U). We shall call such a finite disjoint collection of E_i's a decomposition of E with respect to U and denote it by $\{E_1, E_2, \cdots, E_n \colon U\}$.

Clearly, if μ has the property (*), it can have no atoms.

THEOREM 6. *If a measure μ:$\mathbf{S} \to G$ has the property (*) and $\mathcal{U}(e)$ has a countable base, then the associated uniform space is connected.*

Proof. Suppose that $\mathbf{S}$ is disconnected: i. e., there are disjoint nonempty open subsets $\mathcal{H}$ and $\mathcal{G}$ of $\mathbf{S}$, whose union is $\mathbf{S}$. The empty set $\phi \in \mathbf{S}$ belongs to one of sets $\mathcal{H}$ or else $\mathcal{G}$, say $\mathcal{H}$. Then, there exists a set E in $\mathcal{G}$ with $\mu(E) \neq e$. Let $\mathcal{B} = \{U_i\}$ be a base as in the proof of Theorem 5. Moreover, we may assume $\tilde{U}_1[\phi] \in \mathcal{H}$ and $\tilde{U}_1[E] \in \mathcal{G}$.

Write $V_1 = U_1$ and let V_2 be an element in $\mathcal{B}$ with $V_2^2 \subset V_1$. If $\mathcal{D} = \{E_1, E_2, \cdots, E_n \colon V_2\}$ is a decomposition of E, there exists an integer $t\,(n > t \geqq 1)$ and $E_{i_p} \in \mathcal{D}$ such that $E - \bigcup_{p=1}^{t} E_{i_p} \in \mathcal{G}$ but $(E - \bigcup_{p=1}^{t} E_{i_p}) - E_k \bar{\in} \mathcal{G}$ for some E_k with $k \neq i_p$ $1 \leq p \leq t$. To see this, note that $E - E_i \in \mathcal{G}$ and $E_i \in \mathcal{H}$ for any i.

Let $B_1 = E$, $B_2 = E - \bigcup_{p=1}^{t} E_{i_p}$, $R_1 = E$ and $R_2 = E_k$ where $B_2 - E_k \bar{\in} \mathcal{G}$. By an induction we shall construct decreasing sequences $\{B_n\}$, $\{R_n\}$ in $\mathbf{S}$ and $\{V_n\}$ in $\mathcal{B}$ such that;

(a) $B_n \in \mathcal{G}$,

(b) $B_n - R_n \in \mathcal{G}$

(c) $\mu(R_n) \in V_n$,

(d) $\tilde{V}_n[B_{n-1}] \subset \mathcal{G}$.

For $n=2$, it has been done in preceding paragraph.

There is V_{n+1} in $\mathcal{B}$ such that $V_{n+1} \subset U_{n+1}$, $V_{n+1}^2 \subset V_n$ and $\tilde{V}_{n+1}[B_n] \subset \mathcal{G}$ because $B_n \in \mathcal{G}$ (the induction hypothesis) and $\mathcal{G}$ is open. Let $\{D_1, D_2, \cdots, D_m \colon V_{n+1}\}$ be a decomposition of R_n. Then, as in case for $n=2$, there is an integer $t\,(m > t \geq 1)$ and r such that $B_n - \bigcup_{p=1}^{t} D_{i_p} \in \mathcal{G}$ but $(B_n - \bigcup_{p=1}^{t} D_{i_p}) - D_k \bar{\in} \mathcal{G}$, $k \neq i_p$ $(1 \leqq p \leqq t)$. The sets $B_n - \bigcup_{p=1}^{t} D_{i_p}$

and D_k may be chosen for B_{n+1} and R_{n+1}, respectively, which completes the induction on n. We note that in the above construction, $B_n - B_{n+1} \subset R_n$ if $n \geqslant 2$.

To see that $\{B_n\}$ and $\{B_n - R_n\}$ are Cauchy sequences in $\mathcal{G}$ and $\mathcal{H}$, respectively, let U be any neighborhood of e. Reverting to the construction of $\{V_n\}$, we can find an integer N such that $n \geqq N$ implies $V_n \subset U$. Therefore, we have $B_n \triangle B_{n+p} = B_n - B_{n+p} \subset R_n$, so that by (c), $\mu(B_n \triangle B_{n+p}) \in U$ for $n \geqq N$. Thus, $\{B_n\}$ is a Cuachy sequence in $\mathcal{G}$. Note that the sequence $\{B_n - R_n\}$ $(n \geqslant 2)$ is increasing and $(B_n - B_n) \triangle (B_{n+p} - R_{n+p}) \subset R_n$, from which it is clear that $\{B_n - R_n\}$ is a Cauchy sequence in $\mathcal{H}$.

Since $\mathbf{S}$ is complete by Theorem 5, the sequences $\{B_n\}$ and $\{B_n - R_n\}$ being arbitrarily near as n tends to infinity, converges to a common limit. This contradicts to the choice of the sets $\mathcal{G}$ and $\mathcal{H}$, completing the proof.

Let S be a locally compact Hausdorff space and $\mathbf{B}$ be the σ-ring of Borel sets. A measure $\mu:\mathbf{B} \to G$ is said to be *regular at* E if, for each $U \in \mathcal{U}(e)$, there exsist a compact set C and an open set H of X such that $C \subset E \subset H$ and $\mu(H-C) \in U$.

LEMMA 7. *If a measure $\mu:\mathbf{B} \to G$ is regular at an atom A, there exists a compact set C such that $A \supset C$ and $\mu(C) = \mu(A)$.*

Proof. Let U be a neighborhood of the identity $e \in G$ with $\mu(A) \notin U$. The regularity of μ at A implies that there exist a compact set C and an open set H such that $C \subset A \subset H$ and $\mu(H-C) \in U$. Since $\mu(H) = \mu(H-C) \cdot \mu(C)$, $\mu(C) = e$ implies $\mu(A) \in U$ which is impossible.

THEOREM 8. *If $\mu:\mathbf{B} \to G$ is a regular measure, then for every atom A there exists a point x in A such that $\mu(A) = \mu(\{x\})$.*

Proof. Let $\mathbf{C}$ be the collection of all compact subsets C such that $A \supset C$ and $\mu(C) = \mu(A)$. Then, by Lemma 7, $\mathbf{C}$ is not empty. If C_1 and C_2 are in $\mathbf{C}$, then $C_1 \cap C_2$ is in $\mathbf{C}$. Otherwise, either $\mu(C_1) = e$ or $\mu(C_2) = e$. It follows that $\mathbf{C}$ has finite intersection property, and hence $F = \bigcap \{C: C \in \mathbf{C}\}$ is not empty. We shall show that F consists of single point x. If there were another point $y \neq x$, let H be any open set containing x but not y. Let C be any set in $\mathbf{C}$, then either $\mu(C \cap H) = \mu(A)$ or $\mu(C-H) = \mu(A)$. Reverting to Lemma 7, if $\mu(C \cap H) = \mu(A)$, there exists a compact set $K \subset C \cap H$ with $\mu(K) = \mu(A)$. This means $y \in F$, leading to a contradiction. If $\mu(C-H) = \mu(A)$, then $C-H$ being compact is in $\mathbf{C}$. This is impossible since $x \in C-H$. Thus $F = \{x\} \subset A$.

Now, it remains only to prove $\mu(\{x\}) \neq e$. Let U be any neighborhood of e such that $\mu(A) \notin U$. Then $\mu(H-x) \in U$ for some open set H containing x since μ is regular at $\{x\}$. Since every set in $\mathbf{C}$ is compact, there exist finite number of sets C_i in $\mathbf{C}$ such that $\bigcap_{i=1}^{n} C_i \subset H$ and $\mu(\bigcap_{i=1}^{n} C_i) = \mu(A)$. If it were $\mu(\{x\}) = e$, then $\mu(H) = \mu(H-x) \cdot \mu(x) = \mu(H-x) \in U$. The monotonity of μ implies that $\mu(\bigcap_{i=1}^{n} C_i) \in U$, which contradicts our assumption. This completes the proof.

References

[1] D. Landers, *Connectedness properties of the range of vector and semi measures*, Manuscripta Math. 9(1973), 105-112.

[2] K. Sundaresan and P. W. Days, *Regularity of group valued Bair and Borel measures*,

Proc. Amer. Math. Soc. 36(1972), 609–612.
[3] P. R. Halmos, *Measure theorey*, Van Nostrand, New York, 1950.

Seoul National University