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AFFINE CONNECTIONS IN A NORMAL (ϕ, ψ)-MANIFOLD WITH COMPLEMENTED FRAMES

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0. Introduction.

Let M be an N-dimensional differentiable manifold of class C^∞ and let there be given a tensor field f of type $(1,1)$ satisfying $f^3+f=0$. If there exist $N-r$ vector fields U_a $(a=1,2,\cdots, N-r)$ spanning the distribution corresponding to f^2+1 and $N-r$ 1-forms u^a satisfying

$$f^2=-1+\sum_{a=1}^{N-r}u^a\otimes U_a,$$

$$fU_a=0,\quad u^a\circ f=0,\quad u^a(U_b)=\delta_b{}^a,$$

$(a, b=1,2,\cdots, N-r)$, where $\delta_b{}^a$ is the Kronecker delta, then the set (f, U_a, u^a) is called an f-structure with complemented frames and M an f-manifold with complemented frames.

The f-manifold with complemented frames such that $r=N-2$ has been studied by K. Yano [4].

In the present paper, we consider f-structures with complemented frames such that $r=N-3$. In §1, we show that such a structure induces three almost contact structures and in §2, we define and study the normality of such a structure. In §3, we study the cosymplectic property on such a structure. In §4, we study affine connections on such a structure.

In §5, we investigate a (ϕ, ψ)-structure with complemented frames which is defined by two f-structures with complemented frames and in §6, we define and study the normality of such a structure. In §7, we study a symmetric affine connection in a normal (ϕ, ψ)-manifold with complemented frames.

1. f-structure with complemented frames.

Let M be a $(4n+3)$-dimensional differentiable manifold and let there be given a tensor field f of type $(1,1)$ and of rank $4n$, three vector fields U, V, W and three 1-forms u, v, w.

If the set (f, U, V, W, u, v, w) satisfies

$$f^2=-1+u\otimes U+v\otimes V+w\otimes W, \tag{1. 1}$$

$$fU=fV=fW=0,\quad u\circ f=v\circ f=w\circ f=0, \tag{1. 2}$$

$$\begin{gathered}u(U)=v(V)=w(W)=1,\\ u(V)=u(W)=0,\quad v(U)=v(W)=0,\quad w(U)=w(V)=0,\end{gathered} \tag{1. 3}$$

then (f, U, V, W, u, v, w) is called an f-structure with complemented frames and M an f-manifold with complemented frames [4].

We define a tensor field F of type $(1,1)$ by

$$FX=fX-w(X)V+v(X)W \tag{1. 4}$$

for an arbitrary vector field X. Then we have

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$$F^2X=f^2X-v(X)V-w(X)W,$$

that is,

$$F^2=-1+u\otimes U. \tag{1.5}$$

Similarly, we define two tensor fields $'F$ and $''F$ of type $(1, 1)$ by

$$\begin{aligned} 'FX&=fX-u(X)W+w(X)U,\\ ''FX&=fX-v(X)U+u(X)V \end{aligned} \tag{1.6}$$

respectively, we find

$$\begin{aligned} 'F^2&=-1+v\otimes V,\\ ''F^2&=-1+w\otimes W. \end{aligned} \tag{1.7}$$

We can easily verify that

$$\begin{array}{lll} FU=0, & FV=W, & FW=-V,\\ 'FU=-W, & 'FV=0, & 'FW=U,\\ ''FU=V, & ''FV=-U, & ''FW=0, \end{array} \tag{1.8}$$

and

$$\begin{array}{lll} u\circ F=0, & v\circ F=-w, & w\circ F=v,\\ u\circ 'F=w, & v\circ 'F=0, & w\circ 'F=-u,\\ u\circ ''F=-v, & v\circ ''F=u, & w\circ ''F=0, \end{array} \tag{1.9}$$

and consequently (F, U, u), $('F, V, v)$ and $(''F, W, w)$ are three almost contact structures.

Conversely, suppose that a $(4n+3)$-dimensional differentiable manifold M admits three almost contact structures (F, U, u), $('F, V, v)$ and $(''F, W, w)$ satisfying (1.8) and (1.9).

We define a tensor field f of type $(1, 1)$ by

(1.10) $\qquad fX= FX+w(X)V-v(X)W,$

or $\qquad fX= 'FX+u(X)W-w(X)U,$

or $\qquad fX= ''FX+v(X)U-u(X)V,$

then we can easily verify that

$$f^2=-1+u\otimes U+v\otimes V+w\otimes W$$

and

$$fU=fV=fW=0,\quad u\circ f=v\circ f=w\circ f=0.$$

Thus, we have

THEOREM 1.1. *In order that a manifold admits an f-structure with complemented frames (f, U, V, W, u, v, w), it is necessary and sufficient that the manifold admits three almost contact structures (F, U, u), $('F, V, v)$ and $(''F, W, w)$ satisfying* (1.8) *and* (1.9).

2. Normal f-structure with complemented frames.

For an f-structure with complemented frames (f, U, V, W, u, v, w) satisfying (1.1), (1.2) and (1.3), we define a tensor field S of type $(1, 2)$ by

$$S(X, Y)=\frac{1}{2}[f, f](X, Y)+du(X, Y)U+dv(X, Y)V+dw(X, Y)W, \tag{2.1}$$

where $[f, f]$ is the Nijenhuis tensor formed with f, that is,

$$[f, f](X, Y)=2\{[fX, fX]-f[fX, Y]-f[X, fY]+f^2[X, Y]\}. \tag{2.2}$$

If the tensor field S vanishes identically, then the structure is said to be normal.

We now assume that the f-structure with complemented frames (f, U, V, W, u, v, w) is normal, then by the same process to K. Yano's theorem 1.1 of [4], we have the following

THEOREM 2.1. *If an f-structure with complemented frames* (f, U, V, W, u, v, w) *is normal, then we have*

$$
\begin{aligned}
&\mathcal{L}_U f=0, \quad \mathcal{L}_U u=0, \quad \mathcal{L}_U v=0, \quad \mathcal{L}_U w=0,\\
&\mathcal{L}_V f=0, \quad \mathcal{L}_V u=0, \quad \mathcal{L}_V v=0, \quad \mathcal{L}_V w=0,\\
&\mathcal{L}_W f=0, \quad \mathcal{L}_W u=0, \quad \mathcal{L}_W v=0, \quad \mathcal{L}_W w=0,\\
&du\barwedge f=0, \quad dv\barwedge f=0, \quad dw\barwedge f=0,\\
&[U, V]=0, \quad [V, W]=0, \quad [W, U]=0,
\end{aligned} \tag{2.3}
$$

where $\mathcal{L}$ *denotes the operation of the Lie differentiation and symbol* $\barwedge$ *is defined by*

$$(\omega\barwedge f)(X, Y)=\omega(fX, Y)+\omega(X, fY)$$

for a 2-form ω [4].

Next, for three almost contact structures (F, U, u), $('F, V, v)$ and $(''F, W, w)$, we define three tensor fields N, $'N$ and $''N$ of type $(1, 2)$ respectively by

$$N(X, Y)=\frac{1}{2}[F, F](X, Y)+du(X, Y)U,$$

$$'N(X, Y)=\frac{1}{2}['F, 'F](X, Y)+dv(X, Y)V,$$

$$''N(X, Y)=\frac{1}{2}[''F, ''F](X, Y)+dw(X, Y)W,$$

where $[F, F]$, $['F, 'F]$ and $[''F, ''F]$ are the Nijenhuis tensors formed with F, $'F$ and $''F$ respectively.

By the definition (1.4) of F, we have

$$
\begin{aligned}
N(X, Y)=&[FX, FY]-F[FX, Y]-F[X, FY]+F^2[X, Y]+du(X, Y)U\\
=&[fX-w(X)V+v(X)W,\ fY-w(Y)V+v(Y)W]\\
&-f[fX-w(X)V+v(X)W,\ Y]+w[fX-w(X)V+v(X)W,\ Y]V\\
&-v[fX-w(X)V+v(X)W,\ Y]W-f[X,\ fY-w(Y)V+v(Y)W]\\
&+w[X,\ fY-w(Y)V+v(Y)W]V-v[X,\ fY-w(Y)V+v(Y)W]W\\
&-[X, Y]+u[X, Y]U+du(X, Y)U,
\end{aligned}
$$

from which, after some calculations,

$$
\begin{aligned}
N(X, Y)=&\frac{1}{2}[f, f](X, Y)+du(X, Y)U+dv(X, Y)V+dw(X, Y)W\\
&-(dw\barwedge f)(X, Y)V+(dv\barwedge f)(X, Y)W\\
&-w(X)(\mathcal{L}_V f)Y+w(Y)(\mathcal{L}_V f)X+v(X)(\mathcal{L}_W f)Y-v(Y)(\mathcal{L}_W f)X\\
&+(w(X)(\mathcal{L}_V w)Y-w(Y)(\mathcal{L}_V w)X-v(X)(\mathcal{L}_W w)Y+v(Y)(\mathcal{L}_W w)X)V\\
&+(v(X)(\mathcal{L}_W v)Y-v(Y)(\mathcal{L}_W v)X-w(X)(\mathcal{L}_V v)Y+w(Y)(\mathcal{L}_V v)X)W\\
&+(v(X)w(Y)-v(Y)w(X))[V, W].
\end{aligned}
$$

Similar expressions for $'N(X, Y)$ and $''N(X, Y)$ are also obtained.

Thus, we have

THEOREM 2.2. *If an f-structure with complemented frames* (f, U, V, W, u, v, w) *is normal, then three almost contact structures* (F, U, u), $('F, V, v)$ *and* $(''F, W, w)$ *defined by* (1.4) *and* (1.6) *are all normal.*

Next, we define the components $[\alpha, \alpha]_{ji}{}^h$ of $[\alpha, \alpha]$ by

$$\frac{1}{2}[\alpha, \alpha]_{ji}{}^h=\alpha_j{}^t\nabla_t\alpha_i{}^h-\alpha_i{}^t\nabla_t\alpha_j{}^h-(\nabla_j\alpha_i{}^t-\nabla_i\alpha_j{}^t)\alpha_t{}^h,$$

where

$$\alpha=u\otimes U+v\otimes V+w\otimes W.$$

By the direct computations, we have

$$\frac{1}{2}[\alpha,\alpha](X,Y)=\{u(X)(\mathcal{L}_U u)Y+v(X)(\mathcal{L}_V u)Y+w(X)(\mathcal{L}_W u)Y$$
$$-u(Y)(\mathcal{L}_U u)X-v(Y)(\mathcal{L}_V u)X-w(Y)(\mathcal{L}_W u)X\}U$$
$$+\{u(X)(\mathcal{L}_U v)Y+v(X)(\mathcal{L}_V v)Y+w(X)(\mathcal{L}_W v)Y$$
$$-u(Y)(\mathcal{L}_U v)X-v(Y)(\mathcal{L}_V v)X-w(Y)(\mathcal{L}_W v)X\}V$$
$$+\{u(X)(\mathcal{L}_U w)Y+v(X)(\mathcal{L}_V w)Y+w(X)(\mathcal{L}_W w)Y$$
$$-u(Y)(\mathcal{L}_U w)X-v(Y)(\mathcal{L}_V w)X-w(Y)(\mathcal{L}_W w)X\}W$$
$$+\{u(X)v(Y)-u(Y)v(X)\}[U,V]$$
$$+\{v(X)w(Y)-v(Y)w(X)\}[V,W]$$
$$+\{w(X)u(Y)-w(Y)u(X)\}[W,U]$$
$$-du(X,Y)U-dv(X,Y)V-dw(X,Y)W.$$

Taking account of theorem 2.1, we have

THEOREM 2.3. *If an f-structure with complemented frames* (f,U,V,W,u,v,w) *is normal, then we have*

$$\frac{1}{2}[\alpha,\alpha]+du\otimes U+dv\otimes V+dw\otimes W=0,$$

where

$$\alpha=u\otimes U+v\otimes V+w\otimes W.$$

3. Cosymplectic f-structure with complemented frames.

Let M be a $(4n+3)$-dimensional differentiable manifold with f-structure with complemented frames (f,U,V,W,u,v,w). If there exists on M a Riemannian metric g satisfying

$$g(fX,fY)=g(X,Y)-u(X)u(Y)-v(X)v(Y)-w(X)w(Y),$$
$$u(X)=g(U,X),\quad v(X)=g(V,X),\quad w(X)=g(W,X) \tag{3. 1}$$

for arbitrary vector fields X and Y, we call the structure (f,U,V,W,u,v,w) a metric f-structure with complemented frames and denote it by (f,g,u,v,w).

Suppose that M admits an (f,g,u,v,w)-structure. We know that M admits three almost contact structures (F,U,u), $('F,V,v)$ and $(''F,W,w)$ defined by (1.4) and (1.6). For these three structures, we have

$$g(FX,FY)=g(fX-w(X)V+v(X)W,\ fY-w(Y)V+v(Y)W)$$
$$=g(fX,fY)+v(X)v(Y)+w(X)w(Y),$$

and consequently

$$g(FX,FY)=g(X,Y)-u(X)u(Y). \tag{3. 2}$$

Similarly, we have

$$g('FX,'FY)=g(X,Y)-v(X)v(Y),$$
$$g(''FX,''FY)=g(X,Y)-w(X)w(Y). \tag{3. 3}$$

Thus, we obtain three almost contact metric structures (F,g,u), $('F,g,v)$ and $(''F,g,w)$.

Conversely, if M admits three almost contact metric structures, we have, by virtue of (1.4) and (1.9),

$$g(fX,fY)=g(FX+w(X)V-v(X)W,\ FY+w(Y)V-v(Y)W)$$
$$=g(FX,FY)-w(X)w(Y)-v(X)v(Y),$$

and consequently

$$g(fX,fY)=g(X,Y)-u(X)u(Y)-v(X)v(Y)-w(X)w(Y).$$

Thus, we have

THEOREM 3.1. *In order that a manifold admits a metric f-structure with complemented frames (f, g, u, v, w), it is necessary and sufficient that the manifold admits three almost contact metric structures (F, g, u), $('F, g, v)$ and $(''F, g, w)$ satisfying* (1.8) *and* (1.9).

For a metric f-manifold with complemented frames $M(f, g, u, v, w)$, a 2-form l called the fundamental form of M is defined by

(3. 4) $$l(X, Y)=g(fX, Y).$$

If the fundamental form is closed and three 1-forms u, v, w are all closed, that is,

(3. 5) $$dl=0,\ du=0,\ dv=0,\ dw=0,$$

we call M an almost cosymplectic f-manifold with complemented frames. Moreover, if M is almost cosymplectic, so the normality condition is given by the vanishing of the tensor $[f, f]$, and in this case, we call M is cosymplectic.

It is well known that if an almost contact metric manifold $M(F, g, u)$ is cosymplectic, then both $\overset{0}{\nabla}F$, $\overset{0}{\nabla}u$ vanish, where $\overset{0}{\nabla}$ denotes covariant differentiation with respect to the Riemannian connection [1].

It would be natural to ask if an f-manifold with complemented frames $M(f, g, u, v, w)$ is cosymplectic, then $\overset{0}{\nabla}f$, $\overset{0}{\nabla}u$, $\overset{0}{\nabla}v$ and $\overset{0}{\nabla}w$ all vanish.

For the almost contact metric structure (F, g, u), we define the fundamental 2-form by

$$L(X, Y)=g(FX, Y).$$

From (1.4) we get

$$l(X, Y)=L(X, Y)+w(X)v(Y)-v(X)w(Y),$$

and from which, we find

$$\begin{aligned} dl(X, Y, Z)=dL(X, Y, Z)&+w(X)dv(Y, Z)-v(X)dw(Y, Z)\\ &+w(Y)dv(Z, X)-v(Y)dw(Z, X)\\ &+w(Z)dv(X, Y)-v(Z)dw(X, Y). \end{aligned}$$

Suppose that the metric f-structure with complemented frames (f, g, u, v, w) is almost cosymplectic. Then we have $dL=0$ by virtue of (3.5), and in this case, the almost contact metric structure (F, g, u) is also almost cosymplectic [1].

Similarly, under the same assumption, we can conclude that the almost contact metric structures $('F, g, v)$ $(''F, g, w)$ are both almost cosymplectic. Thus, by theorem 2.2, we can conclude that if the metric f-structure with complemented frames (f, g, u, v, w) is cosymplectic, then three almost contact metric structures (F, g, u), $('F, g, v)$ and $(''F, g, w)$ are all cosymplectic, and consequently we have

$$\overset{0}{\nabla}F=0,\ \overset{0}{\nabla}u=0,\ \overset{0}{\nabla}'F=0,\ \overset{0}{\nabla}v=0,\ \overset{0}{\nabla}''F=0,\ \overset{0}{\nabla}w=0.$$

In this case, we have from (1.10) $\overset{0}{\nabla}f=0$, and from which, we have $\overset{0}{\nabla}U=0$, $\overset{0}{\nabla}V=0$ and $\overset{0}{\nabla}W=0$ by virtue of (1.2).

Thus, we have

THEOREM 3.2. *If a metric f-manifold with complemented frames $M(f, g, u, v, w)$ is cosymplectic, then $\overset{0}{\nabla}f$, $\overset{0}{\nabla}U$, $\overset{0}{\nabla}V$, $\overset{0}{\nabla}W$, $\overset{0}{\nabla}u$, $\overset{0}{\nabla}v$ and $\overset{0}{\nabla}w$ all vanish.*

4. Affine connections in an f-manifold with complemented frames.

S. Ishihara and M. Obata [2] have proved following

THEOREM A. *In order that a manifold M admits an affine connection without torsion which has a set of p independent non-zero parallel vector fields, it is necessary and* *sufficient that M admits a commutative ρ-frame.*

In theorem A, a commutative ρ-frame is defined by a set of independent ρ vector fields U_a such that

$$[U_a, U_b]=0, \quad (a, b=1, 2, \cdots, \rho).$$

On a normal f-manifold with complemented frames M, we have, from theorem 2.1,

$$[U, V]=0, \qquad [V, W]=0, \qquad [W, U]=0$$

and U, V, W are independent, therefore the following Lemma is an immediate consequence of the theorem A.

LEMMA 4.1. *A normal f-manifold with complemented frames $M(f, U, V, W, u, v, w)$ admits a symmetric affine connection $\overset{0}{\nabla}$ such that $\overset{0}{\nabla}U=0$, $\overset{0}{\nabla}V=0$ and $\overset{0}{\nabla}W=0$.*

Next, we suppose that in a normal f-manifold with complemented frames $M(f, U, V, W, u, v, w)$, three 1-forms u, v, w are all closed, or equivalently the Nijenhuis tensor $[f, f]$ formed with f vanishes.

We denote the components of the connection $\overset{0}{\nabla}$ such that $\overset{0}{\nabla}U=0$, $\overset{0}{\nabla}V=0$ and $\overset{0}{\nabla}W=0$ by $\overset{0}{\Gamma}{}_{kj}^{h}$ and put

$$\overset{1}{\Gamma}{}_{kj}^{h}=\overset{0}{\Gamma}{}_{kj}^{h}+U^h\overset{0}{\nabla}_k u_j+V^h\overset{0}{\nabla}_k v_j+W^h\overset{0}{\nabla}_k w_j. \tag{4.1}$$

Then denoting by $\overset{1}{\nabla}_j$ the operator of covariant differentiation with respect to $\overset{1}{\Gamma}{}_{kj}^{h}$, we see that

$$\begin{aligned}\overset{1}{\nabla}_k U^h&=\overset{0}{\nabla}_k U^h+(U^h\overset{0}{\nabla}_k u_j+V^h\overset{0}{\nabla}_k v_j+W^h\overset{0}{\nabla}_k w_j)U^j\\&=-(\overset{0}{\nabla}_k U^j)(U^h u_j+V^h v_j+W^h w_j)\\&=0.\end{aligned} \tag{4.2}$$

Similarly, we also have

$$\overset{1}{\nabla}_k V^h=0, \qquad \overset{1}{\nabla}_k W^h=0.$$

Calculating $\overset{1}{\nabla}_k u_h$, we see that

$$\overset{1}{\nabla}_k u_h=\overset{0}{\nabla}_k u_h-\overset{0}{\nabla}_k u_h=0. \tag{4.3}$$

Similarly, we have

$$\overset{1}{\nabla}_k v_h=0, \qquad \overset{1}{\nabla}_k w_h=0.$$

We next put [3]

$$\overset{2}{\Gamma}{}_{kj}^{h}=\overset{1}{\Gamma}{}_{kj}^{h}+\overset{1}{T}{}_{kj}^{h}, \tag{4.4}$$

where

$$\overset{1}{T}{}_{kj}^{h}=-\frac{1}{4}\{f_j^t(\overset{1}{\nabla}_t f_k^h-\overset{1}{\nabla}_k f_t^h)+f_t^h(\overset{1}{\nabla}_j f_k^t+\overset{1}{\nabla}_k f_j^t)\}. \tag{4.5}$$

Now denoting by $\overset{2}{\nabla}_j$ the operator of covariant differentiation with respect to $\overset{2}{\Gamma}{}_{kj}^{h}$, we see that

$$\begin{aligned}\overset{2}{\nabla}_k f_j^h&=\overset{1}{\nabla}_k f_j^h+\overset{1}{T}{}_{ki}^{h}f_j^i-\overset{1}{T}{}_{kj}^{i}f_i^h\\&=\overset{1}{\nabla}_k f_j^h-\frac{1}{4}\overset{1}{\nabla}_k f_j^h-\frac{1}{4}(\overset{1}{\nabla}_t f_k^h)(u_jU^t+v_jV^t+w_jW^t)-\frac{1}{4}\overset{1}{\nabla}_k f_j^h-\frac{1}{2}\overset{1}{\nabla}_k f_j^h.\end{aligned}$$

Since $\mathcal{L}_U f=0$, $\mathcal{L}_V f=0$ and $\mathcal{L}_W f=0$ by virtue of (2.3), we have

(4. 6) $$U^t \overset{1}{\nabla}_t f_k{}^h=0, \quad V^t \overset{1}{\nabla}_t f_k{}^h=0, \quad W^t \overset{1}{\nabla}_t f_k{}^h=0,$$
and consequently

(4. 7) $$\overset{2}{\nabla}_k f_j{}^h=0.$$

For the covariant derivatives of U^h and u_h with respect to $\overset{2}{\nabla}$, we have

(4. 8) $$\overset{2}{\nabla}_k U^h=\overset{1}{T}_{kt}{}^h U^t=-\frac{1}{4}\overset{1}{\nabla}_k(f_t{}^h U^t)=0,$$

(4. 9) $$\overset{2}{\nabla}_k u_h=-\overset{1}{T}_{kh}{}^t u_t=\frac{1}{4}f_h{}^s\{\overset{1}{\nabla}_s(f_k{}^t u_t)-\overset{1}{\nabla}_k(f_s{}^t u_t)\}=0.$$

Similarly, we have

(4.10) $$\overset{2}{\nabla}_k V^h=0, \qquad \overset{2}{\nabla}_k W^h=0.$$
and

(4.11) $$\overset{2}{\nabla}_k v_h=0, \qquad \overset{2}{\nabla}_k w_h=0$$

Since $\overset{1}{\nabla}$ is a symmetric connection, we have, from (4.4) and (4.5)

$$\begin{aligned}\overset{2}{\Gamma}_{kj}{}^h-\overset{2}{\Gamma}_{jk}{}^h&=\overset{1}{T}_{kj}{}^h-\overset{1}{T}_{jk}{}^h\\ &=\frac{1}{4}\{f_k{}^t(\overset{1}{\nabla}_t f_j{}^h-\overset{1}{\nabla}_j f_t{}^h)-f_j{}^t(\overset{1}{\nabla}_t f_k{}^h-\overset{1}{\nabla}_k f_t{}^h)\\ &=\frac{1}{8}[f,f]_{kj}{}^h,\end{aligned}$$

where $[f,f]_{kj}{}^h$ are components of the Nijenhuis tensor $[f,f]$ formed with f.

Thus, we have

THEOREM 4.2. *On a normal f-manifold with complemented frames $M(f,U,V,W,u,v,w)$, we can find a symmetric affine connection $\overset{2}{\nabla}$ such that $\overset{2}{\nabla}f$, $\overset{2}{\nabla}U$, $\overset{2}{\nabla}V$, $\overset{2}{\nabla}W$, $\overset{2}{\nabla}u$, $\overset{2}{\nabla}v$, $\overset{2}{\nabla}w$ all vanish if and only if one of the following conditions is satisfied:*

i) *Three 1-forms u,v and w are all closed.*

ii) *Nijenhuis tensor formed with f vanishes.*

5. (ϕ,ψ)-structure with complemented frames.

Let M be a $(4n+3)$-dimensional differentiable manifold and let there be given two tensor fields ϕ,ψ of type $(1,1)$ and of rank $4n$, three vector fields U,V,W and three 1-forms u,v,w.

If the set (ϕ,ψ,U,V,W,u,v,w) satisfies

(5. 1) $$\phi^2=-1+u\otimes U+v\otimes V+w\otimes W,$$
$$\psi^2=-1+u\otimes U+v\otimes V+w\otimes W,$$

(5. 2) $$\phi\psi=-\psi\phi,$$

(5. 3) $$\phi U=\phi V=\phi W=0, \qquad \psi U=\psi V=\psi W=0,$$

(5. 4) $$u\circ\phi=v\circ\phi=w\circ\phi=0, \qquad u\circ\psi=v\circ\psi=w\circ\psi=0,$$

(5. 5) $$u(U)=v(V)=w(W)=1,$$
$$u(V)=u(W)=0, \qquad v(W)=v(U)=0, \qquad w(U)=w(V)=0,$$

then we call the set (ϕ,ψ,U,V,W,u,v,w) a (ϕ,ψ)-structure with complemented frames and M a (ϕ,ψ)-manifold with complemented frames. Obviously, (ϕ,U,V,W,u,v,w) and (ψ,U,V,W,u,v,w) are two f-structures with complemented frames defined in §1.

We define a tensor field θ of type $(1,1)$ by

(5. 6) $$\theta=\phi\psi.$$

Then we have

$$\theta^2=-1+u\otimes U+v\otimes V+w\otimes W, \tag{5.7}$$

$$\phi\psi=-\psi\phi=\theta, \qquad \psi\theta=-\theta\psi=\phi, \qquad \theta\phi=-\phi\theta=\psi, \tag{5.8}$$

$$\theta U=\theta V=\theta W=0, \qquad u\circ\theta=v\circ\theta=w\circ\theta=0. \tag{5.9}$$

Thus, $(\theta, U, V, W, u, v, w)$ also an f-structure with complemented frames defined in §1.

Next, we define three tensor fields F, G and H of type $(1,1)$ by

$$\begin{aligned} F&=\phi-w\otimes V+v\otimes W,\\ G&=\psi-u\otimes W+w\otimes U,\\ H&=\theta-v\otimes U+u\otimes V,\end{aligned} \tag{5.10}$$

then from (1.5), (1.7), (1.8) and (1.9) we can easily verify that

$$\begin{aligned} F^2&=-1+u\otimes U,\\ G^2&=-1+v\otimes V,\\ H^2&=-1+w\otimes W,\end{aligned} \tag{5.11}$$

$$\begin{matrix} FU=0, & FV=W, & FW=-V,\\ GU=-W, & GV=0, & GW=U,\\ HU=V, & HV=-U, & HW=0\end{matrix} \tag{5.12}$$

and

$$\begin{matrix} u\circ F=0, & v\circ F=-w, & w\circ F=v,\\ u\circ G=w, & v\circ G=0, & w\circ G=-u,\\ u\circ H=-v, & v\circ H=u, & w\circ H=0.\end{matrix} \tag{5.13}$$

By the definitions (5.10), we also have

$$FGX=\theta X+u(X)V=HX+v(X)U,$$

for arbitrary vector field X of M, that is,

$$FG=H+v\otimes U.$$

Similarly, we have

$$GH=F+w\otimes V, \qquad HF=G+u\otimes W, \qquad GF=-H+u\otimes V,$$
$$FH=-G+w\otimes U, \qquad HG=-F+v\otimes W,$$

and these equations show that M admits an almost contact affine 3-structure $(F, G, H, U, V, W, u, v, w)$ [6].

Conversely, we suppose that M admits an almost contact affine 3-structure $(F, G, H, U, V, W, u, v, w)$.

Defining for an arbitrary vector field X

$$\begin{aligned} \phi X&=FX+w(X)V-v(X)W,\\ \psi X&=GX+u(X)W-w(X)U,\\ \theta X&=HX+v(X)U-u(X)V,\end{aligned} \tag{5.14}$$

we can easily verify that

$$\phi^2=\psi^2=\theta^2=-1+u\otimes U+v\otimes V+w\otimes W,$$
$$\phi\psi=-\psi\phi=\theta, \qquad \psi\theta=-\theta\psi=\phi, \qquad \theta\phi=-\phi\theta=\psi.$$

Moreover, we can obtain the conditions (5.3), (5.4) and (5.9), and consequently these equations show that M admits a (ϕ, ψ)-structure with complemented frames. Thus, we have

THEOREM 5.1. *In order that a manifold admits a (ϕ, ψ)-structure with complemented frames, it is necessary and sufficient that the manifold admits an almost contact affine*

3-structure.

Next, we take an arbitrary Riemannian metric a in (ϕ,ψ)-manifold with complemented frames and put

$$(5.15)\qquad b(X,Y)=a(\phi X,\phi Y)+a(\phi^2X,\phi^2Y)+u(X)u(Y)+v(X)v(Y)+w(X)w(Y)$$

for arbitrary vector fields X, Y of M.

Then we easily see by (5.1), (5.5) and (5.7) that

$$(5.16)\qquad b(\phi X,\phi Y)=a(\phi^2X,\phi^2Y)+a(\phi X,\phi Y),$$

or

$$(5.17)\qquad b(\phi X,\phi Y)=b(X,Y)-u(X)u(Y)-v(X)v(Y)-w(X)w(Y).$$

We next put

$$(5.18)\qquad g(X,Y)=b(X,Y)+b(\psi X,\psi Y),$$

then we find

$$g(\phi X,\phi Y)=b(\phi X,\phi Y)+b(\phi\psi X,\phi\psi Y)$$

or

$$(5.19)\qquad g(\phi X,\phi Y)=b(\phi X,\phi Y)+b(\psi X,\psi Y)$$

by virtue of (5.17).

From (5.17), (5.18) and (5.19), we have

$$(5.20)\qquad g(\phi X,\phi Y)=g(X,Y)-u(X)u(Y)-v(X)v(Y)-w(X)w(Y).$$

From (5.15), we have

$$(5.21)\qquad b(\psi X,\ \psi Y)=a(\theta X,\theta Y)+a(\psi X,\psi Y).$$

Substituting (5.21) into (5.18), we have

$$g(X,Y)=b(X,Y)+a(\theta X,\theta Y)+a(\psi X,\psi Y),$$

from which,

$$g(\theta X,\theta Y)=b(\theta X,\theta Y)+a(\theta^2X,\theta^2Y)+a(\phi X,\phi Y)$$

or

$$(5.22)\qquad g(\theta X,\theta Y)=b(\theta X,\theta Y)+b(X,Y)-u(X)u(Y)-v(X)v(Y)-w(X)w(Y)$$

by virtue of (5.16) and (5.17).

Replacing X and Y by ϕX and ϕY respectively in (5.22), we find

$$(5.23)\qquad g(\psi X,\psi Y)=b(\psi X,\psi Y)+b(\phi X,\phi Y).$$

Comparing (5.19) with (5.23), we see that

$$(5.24)\qquad g(\psi X,\psi Y)=g(\phi X,\phi Y).$$

Replacing X and Y by ψX and ψY respectively in (5.20), we find

$$(5.25)\qquad g(\theta X,\theta Y)=g(\psi X,\psi Y).$$

Equations (5.20), (5.24) and (5.25) show that

$$(5.26)\qquad \begin{aligned} g(\psi X,\psi Y)&=g(X,Y)-u(X)u(Y)-v(X)v(Y)-w(X)w(Y),\\ g(\theta X,\theta Y)&=g(X,Y)-u(X)u(Y)-v(X)v(Y)-w(X)w(Y). \end{aligned}$$

Next, replacing Y by U in (5.20) and taking account of (5.5), we find

$$(5.27)\qquad u(X)=g(U,X).$$

Similarly, we have

$$(5.28)\qquad v(X)=g(V,X),\qquad w(X)=g(W,X).$$

Thus, we have

THEOREM 5.2. *There exists a Riemannian metric g satisfying* (5.20), (5.26), (5.27) *and* (5.28) *in a (ϕ,ψ)-manifold with complemented frames.*

We call a (ϕ,ψ)-structure with complemented frames with a Riemannian metric g satisfying (5.20), (5.26), (5.27) and (5.28) a metric (ϕ,ψ)-structure with complemented frames.

Substituting the first equation of (5.14) into (5.20), we find

$$g(FX, FY)=g(X, Y)-u(X)u(Y) \tag{5.29}$$

by virtue of

$$g(FX, V)=v(FX)=-w(X), \qquad g(FX, W)=w(FX)=v(X).$$

Similarly, we have

$$\begin{aligned} g(GX, GY)&=g(X, Y)-v(X)v(Y),\\ g(HX, HY)&=g(X, Y)-w(X)w(Y). \end{aligned} \tag{5.30}$$

Equations (5.29) and (5.30) show that M admits an almost contact metric 3-structure [6].

Conversely, if M admits an almost contact metric 3-structure (F, G, H, g, u, v, w), then substituting the first equation of (5.10) into (5.29), we find

$$g(\phi X, \phi Y)=g(FX, FY)-v(X)v(Y)-w(X)w(Y),$$

that is,

$$g(\phi X, \phi Y)=g(X, Y)-u(X)u(Y)-v(X)v(Y)-w(X)w(Y).$$

Similarly, we can obtain (5.26).

Thus, we have

THEOREM 5.3. *In order that a manifold admits a metric (ϕ,ψ)-structure with complemented frames, it is necessary and sufficient that the manifold admits an almost contact metric 3-structure.*

6. Normal (ϕ,ψ)-structure with complemented frames.

On a (ϕ,ψ)-manifold with complemented frames M, if structures (ϕ, U, V, W, u, v, w) and (ψ, U, V, W, u, v, w) are both normal, that is,

$$\begin{aligned} &\frac{1}{2}[\phi,\phi]+du\otimes U+dv\otimes V+dw\otimes W=0,\\ &\frac{1}{2}[\psi,\psi]+du\otimes U+dv\otimes V+dw\otimes W=0, \end{aligned} \tag{6.1}$$

where $[\phi,\phi]$ and $[\psi,\psi]$ are the Nijenhuis tensors formed with ϕ and ψ respectively, then we call M a normal (ϕ,ψ)-manifold with complemented frames.

Putting

$$\Sigma(X, Y)=du(X, Y)U+dv(X, Y)V+dw(X, Y)W$$

for arbitrary vector fields X, Y of M, (6.1) can be written by

$$[\phi,\phi]+2\Sigma=0, \qquad [\psi,\psi]+2\Sigma=0, \tag{6.2}$$

Σ being a tensor field of type (1, 2).

Let P and Q be two tensor fields of type (1, 1) in a differentiable manifold. It is well known that the expression given by

$$\begin{aligned} [P,Q](X, Y)=&[PX, QY]-P[QX, Y]-Q[X, PY]\\ &+[QX, PY]-Q[PX, Y]-P[X, QY]+(PQ+QP)[X, Y], \end{aligned} \tag{6.3}$$

defines a tensor field of type (1, 2) and is called the Nijenhuis tensor of P and Q [5].

The components $[P,Q]_{ji}{}^h$ of $[P,Q]$ are given by

$$\begin{aligned} [P,Q]_{ji}{}^h=&P_j{}^t\nabla_t Q_i{}^h-P_i{}^t\nabla_t Q_j{}^h-(\nabla_j P_i{}^t-\nabla_i P_j{}^t)Q_t{}^h\\ &+Q_j{}^t\nabla_t P_i{}^h-Q_i{}^t\nabla_t P_j{}^h-(\nabla_j Q_i{}^t-\nabla_i Q_j{}^t)P_t{}^h. \end{aligned} \tag{6.4}$$

If S is a tensor field of type $(1, 2)$ and N is a tensor field of type $(1, 1)$, then $S \barwedge N$ is defined to be [5]

(6. 5) $$(S \barwedge N)(X, Y) = S(NX, Y) + S(X, NY)$$

and $N \barwedge S$ to be

(6. 6) $$(N \barwedge S)(X, Y) = NS(X, Y).$$

Then we have the following formula [5], for three tensor fields L, M and N of type $(1, 1)$,

(6. 7) $$[L, MN] + [M, LN] = [L, M] \barwedge N + L \barwedge [M, N] + M \barwedge [L, N].$$

First of all, since

$$(\phi \barwedge \Sigma)(X, Y) = \phi(du(X, Y)U + dv(X, Y)V + dw(X, Y)W),$$

we have

$$\phi \barwedge \Sigma = 0.$$

Similarly, we have

$$\psi \barwedge \Sigma = 0, \qquad \theta \barwedge \Sigma = 0.$$

In this section, we suppose that (ϕ, ψ)-structure with complemented frames is normal, that is, two equations of (6.2) are satisfied.

At first, putting $L = M = \phi$, $N = \psi$ in (6.7), we find

(6. 8) $$[\phi, \theta] = \phi \barwedge [\phi, \psi] - \Sigma \barwedge \psi.$$

On the other hand, putting $L = \psi$, $M = N = \phi$ in (6.7), we find

(6. 9) $$[\psi, \alpha] - [\phi, \theta] = [\phi, \psi] \barwedge \phi + \phi \barwedge [\phi, \psi],$$

where α is a tensor field of type $(1, 1)$ defined by

$$\alpha = u \otimes U + v \otimes V + w \otimes W.$$

Adding (6.8) and (6.9), we find

(6.10) $$[\psi, \alpha] = 2\phi \barwedge [\phi, \psi] + [\phi, \psi] \barwedge \phi - \Sigma \barwedge \psi.$$

Now, putting $L = M = \psi$, $N = \phi$ in (6.7), we find

(6.11) $$[\psi, \theta] = -\psi \barwedge [\phi, \psi] + \Sigma \barwedge \phi.$$

On the other hand, putting $L = \phi$, $M = N = \psi$ in (6.7), we find

(6.12) $$[\phi, \alpha] + [\psi, \theta] = [\phi, \psi] \barwedge \psi + \psi \barwedge [\phi, \psi].$$

Subtracting (6.11) from (6.12), we have

(6.13) $$[\phi, \alpha] = [\phi, \psi] \barwedge \psi - \Sigma \barwedge \phi + 2\psi \barwedge [\phi, \psi].$$

Again, putting $L = \theta, M = \phi$, $N = \psi$ in (6.7), we find

(6.14) $$[\theta, \theta] = -2\Sigma + [\phi, \theta] \barwedge \psi + \theta \barwedge [\phi, \psi] + \phi \barwedge [\psi, \theta].$$

On the other hand, putting $L = \theta$, $M = \psi$, $N = \phi$ in (6.7), we find

(6.15) $$[\theta, \theta] = -2\Sigma - [\psi, \theta] \barwedge \phi - \theta \barwedge [\phi, \psi] - \psi \barwedge [\phi, \theta].$$

Subtracting (6.15) from (6.14), we have

(6.16) $$[\phi, \theta] \barwedge \psi + \phi \barwedge [\psi, \theta] + [\psi, \theta] \barwedge \phi + \psi \barwedge [\phi, \theta] + 2\theta \barwedge [\phi, \psi] = 0.$$

Next, putting $L = M = N = \phi$ in (6.7), we find

(6.17) $$[\phi, \alpha] = -\Sigma \barwedge \phi.$$

Similarly, we get

(6.18) $$[\psi, \alpha] = -\Sigma \barwedge \psi, \qquad [\theta, \alpha] = -\Sigma \barwedge \theta.$$

Subtracting (6.17) from (6.13), we find

(6.19) $$[\phi, \psi] \barwedge \psi = -2\psi \barwedge [\phi, \psi].$$

Finally, putting $L = \alpha$, $M = N = \phi$ in (6.7), we find

$$[\alpha, \alpha] = [\alpha, \phi] \barwedge \phi - 2\Sigma + \phi \barwedge [\alpha, \phi]$$

because of $\alpha\barwedge\Sigma=\Sigma$.

In our case, we have, from theorem 2.3, $[\alpha,\alpha]+2\Sigma=0$. Therefore above equation becomes

$$[\alpha,\phi]\barwedge\phi+\phi\barwedge[\alpha,\phi]=0.$$

Similarly, we have

$$[\alpha,\psi]\barwedge\psi+\psi\barwedge[\alpha,\psi]=0,\qquad [\alpha,\theta]\barwedge\theta+\theta\barwedge[\alpha,\theta]=0. \tag{6.20}$$

Combining (6.10) and (6.18), we get

$$[\phi,\psi]\barwedge\phi=-2\phi\barwedge[\phi,\psi]. \tag{6.21}$$

Substituting (6.8) and (6.11) into (6.16), we have

$$\begin{aligned}&(\phi\barwedge[\phi,\psi])\barwedge\psi-(\Sigma\barwedge\psi)\barwedge\psi-\phi\barwedge(\psi\barwedge[\phi,\psi])\\&-(\psi\barwedge[\phi,\psi])\barwedge\phi+(\Sigma\barwedge\phi)\barwedge\phi+\psi\barwedge(\phi\barwedge[\phi,\psi])\\&+2\theta\barwedge[\phi,\psi]=0\end{aligned}$$

by virtue of

$$\begin{aligned}\phi\barwedge(\Sigma\barwedge\phi)&=(\phi\barwedge\Sigma)\barwedge\phi=0,\\ \psi\barwedge(\Sigma\barwedge\psi)&=(\psi\barwedge\Sigma)\barwedge\psi=0.\end{aligned}$$

On the other hand, since

$$(\phi\barwedge[\phi,\psi])\barwedge\psi=\phi\barwedge([\phi,\psi]\barwedge\psi)$$

and

$$\begin{aligned}&-\phi\barwedge(\psi\barwedge[\phi,\psi])+\psi\barwedge(\phi\barwedge[\phi,\psi])\\&\quad=-\phi\psi\barwedge[\phi,\psi]+\psi\phi\barwedge[\phi,\psi]=-2\theta\barwedge[\phi,\psi],\end{aligned}$$

above equation becomes

$$\begin{aligned}&\phi\barwedge([\phi,\psi]\barwedge\psi)-\psi\barwedge([\phi,\psi]\barwedge\phi)\\&=(\Sigma\barwedge\psi)\barwedge\psi-(\Sigma\barwedge\phi)\barwedge\phi.\end{aligned} \tag{6.22}$$

Substituting (6.21) and (6.19) into (6.22), we find

$$-4\theta\barwedge[\phi,\psi]=(\Sigma\barwedge\psi)\barwedge\psi-(\Sigma\barwedge\phi)\barwedge\phi.$$

Transvecting this equation with θ and taking account of

$$\theta((\Sigma\barwedge\psi)\barwedge\psi)=0,\qquad \theta((\Sigma\barwedge\phi)\barwedge\phi)=0,$$

we have

$$[\phi,\psi]=\alpha\barwedge[\phi,\psi],$$

and from which, we find

$$\phi\barwedge[\phi,\psi]=0,\qquad \psi\barwedge[\phi,\psi]=0,\qquad \theta\barwedge[\phi,\psi]=0 \tag{6.23}$$

by virtue of $\phi\alpha=0$, $\psi\alpha=0$ and $\theta\alpha=0$.

From (6.19), (6.21) and (6.23), we have

$$[\phi,\psi]\barwedge\phi=0,\qquad [\phi,\psi]\barwedge\psi=0.$$

Therefore, from (6.8) and (6.11), we get

$$[\phi,\theta]=-\Sigma\barwedge\psi,\qquad [\psi,\theta]=\Sigma\barwedge\phi, \tag{6.24}$$

and from which, we find

$$\phi\barwedge[\phi,\theta]=0,\qquad \phi\barwedge[\psi,\theta]=0 \tag{6.25}$$

by virtue of $\psi\barwedge\Sigma=0$ and $\phi\barwedge\Sigma=0$.

Taking account of (6.18) and (6.24), we find

$$[\psi,\alpha]=[\phi,\theta], \tag{6.26}$$

and from which, we have

$$\phi\barwedge[\psi,\alpha]=\phi\barwedge[\phi,\theta]=0$$

by virtue of (6.25).

Therefore, the first equation of (6.20) becomes

$$[\psi,\alpha]\overline{\wedge}\psi=0. \tag{6.27}$$

Combining (6.26) and (6.27), we find

$$[\phi,\theta]\overline{\wedge}\psi=0. \tag{6.28}$$

Taking account of (6.23), (6.24) and (6.28), from (6.14), we have

$$[\theta,\theta]+2\Sigma=0.$$

In this way, we can prove the following

THEOREM 6.1. *On a (ϕ,ψ)-manifold with complemented frames, if two of the tensors:*

$$\frac{1}{2}[\phi,\phi]+du\otimes U+dv\otimes V+dw\otimes W,$$

$$\frac{1}{2}[\psi,\psi]+du\otimes U+dv\otimes V+dw\otimes W,$$

$$\frac{1}{2}[\theta,\theta]+du\otimes U+dv\otimes V+dw\otimes W$$

vanish, then the other vanishes too.

Next, we assume that $\Sigma=0$, or equivalently,

$$du=0, \qquad dv=0, \qquad dw=0,$$

then the normality condition of a (ϕ,ψ)-structure with complemented frames is given by

$$[\phi,\phi]=0, \qquad [\psi,\psi]=0.$$

In this case, we have

$$[\theta,\theta]=0, \qquad [\phi,\theta]=0, \qquad [\psi,\theta]=0, \tag{6.29}$$

$$[\phi,\alpha]=0, \qquad [\psi,\alpha]=0, \qquad [\theta,\alpha]=0 \tag{6.30}$$

by virtue of (6.17), (6.18) and (6.24).

Putting $L=\phi$, $M=\theta$, $N=\phi$ in (6.7), and taking account of (6.29) and (6.30), we have

$$[\phi,\psi]=0.$$

By the very same way to the theorem of K. Yano and M. Ako [5], we can prove the following

THEOREM 6.2. *On a (ϕ,ψ)-manifold with complemented frames M, we assume that three 1-forms u, v and w are all closed, that is, $du=0$, $dv=0$ and $dw=0$. In this case, if two of six Nijenhuis tensors:*

$$[\phi,\phi],\ [\psi,\psi],\ [\theta,\theta]\ [\psi,\theta],\ [\theta,\phi],\ [\phi,\psi]$$

vanish, then the others vanish too.

7. A symmetric affine connection in a normal (ϕ,ψ)-manifold with complemented frames.

On a normal (ϕ,ψ)-manifold with complemented frames M, we assume that three 1-forms u, v and w are all closed, or equivalently that the Nijenhuis tensor formed with ϕ vanishes. In this case, we can find a symmetric affine connection $\overset{2}{\nabla}$ such that

$$\overset{2}{\nabla}\phi=0, \quad \overset{2}{\nabla}U=0, \quad \overset{2}{\nabla}V=0, \quad \overset{2}{\nabla}W=0,$$

$$\overset{2}{\nabla}u=0, \quad \overset{2}{\nabla}v=0, \quad \overset{2}{\nabla}w=0$$

by virtue of theorem 4.2.

Since M is normal, we have

$$[\psi, \psi]=0, \qquad [\theta, \theta]=0,$$

and from which,

$$\mathcal{L}_U\psi=0, \qquad \mathcal{L}_V\psi=0, \qquad \mathcal{L}_W\psi=0,$$
$$\mathcal{L}_U\theta=0, \qquad \mathcal{L}_V\theta=0, \qquad \mathcal{L}_W\theta=0,$$

that is,

(7. 1)
$$U^t\overset{2}{\nabla}_t\psi_j^h=0, \qquad V^t\overset{2}{\nabla}_t\psi_j^h=0, \qquad W^t\overset{2}{\nabla}_t\psi_j^h=0,$$
$$U^t\overset{2}{\nabla}_t\theta_j^h=0, \qquad V^t\overset{2}{\nabla}_t\theta_j^h=0, \qquad W^t\overset{2}{\nabla}_t\theta_j^h=0.$$

Therefore we have

(7. 2)
$$\psi_a^t(\overset{2}{\nabla}_t\psi_j^h)\psi_k^a=-\overset{2}{\nabla}_k\psi_j^h,$$
$$\psi_a^h(\overset{2}{\nabla}_k\psi_j^t)\psi_t^a=-\overset{2}{\nabla}_k\psi_j^h,$$
$$\psi_i^a(\overset{2}{\nabla}_k\psi_j^h)\psi_a^j=-\overset{2}{\nabla}_k\psi_i^h$$

and

(7. 3)
$$\theta_a^t(\overset{2}{\nabla}_t\theta_j^h)\theta_k^a=-\overset{2}{\nabla}_k\theta_j^h,$$
$$\theta_a^h(\overset{2}{\nabla}_k\theta_j^t)\theta_t^a=-\overset{2}{\nabla}_k\theta_j^h,$$
$$\theta_i^a(\overset{2}{\nabla}_k\theta_j^h)\theta_a^j=-\overset{2}{\nabla}_k\theta_i^h.$$

We put

(7. 4)
$$\Gamma_{kj}^h=\overset{2}{\Gamma}_{kj}^h+T_{kj}^h,$$

where

(7. 5)
$$T_{kj}^h=-\frac{1}{4}\{\psi_j^t\overset{2}{\nabla}_t\psi_k^h+(\overset{2}{\nabla}_k\psi_j^t)\psi_t^h+\theta_j^t\overset{2}{\nabla}_t\theta_k^h+(\overset{2}{\nabla}_k\theta_j^t)\theta_t^h\}$$
$$+\frac{1}{4}(\phi_j^s\overset{2}{\nabla}_s\psi_k^t+\phi_s^t\overset{2}{\nabla}_k\psi_j^s)\theta_t^h-\frac{1}{4}(\overset{2}{\nabla}_k\psi_j^t+\overset{2}{\nabla}_j\psi_k^t)\psi_t^h.$$

Now denoting ∇_j the operator of covariant differentiation with respect to Γ_{kj}^h and following the very same way to the theorem 4.1 of K. Yano and M. Ako [5], we can prove by the reason of conditions (7.2) and (7.3) that

(7. 6)
$$\nabla_k\phi_j^h=0, \qquad \nabla_k\psi_j^h=0, \qquad \nabla_k\theta_j^h=0.$$

For the covariant derivative of U^h with respect to ∇, we have

$$\nabla_kU^h=\overset{2}{\nabla}_kU^h+T_{ki}^hU^i=-\frac{1}{2}(\overset{2}{\nabla}_k\psi_i^t)\psi_t^hU^i-\frac{1}{4}(\overset{2}{\nabla}_i\psi_k^t)\psi_t^hU^i,$$

that is,

(7. 7)
$$\nabla_kU^h=0$$

by virtue of (7.1).

Similarly, we have

(7. 8)
$$\nabla_kV^h=0, \qquad \nabla_kW^h=0.$$

For the covariant derivative of u_h with respect to ∇, we have

$$\nabla_ku_h=\overset{2}{\nabla}_ku_h-T_{kh}^iu_i=-\frac{1}{4}(\psi_h^t\overset{2}{\nabla}_t\psi_k^i+\theta_h^t\overset{2}{\nabla}_t\theta_k^i)u_i,$$

that is,

(7. 9)
$$\nabla_ku_h=0.$$

Similarly, we have

(7. 10)
$$\nabla_kv_h=0, \qquad \nabla_kw_h=0.$$

Since $\overset{2}{\nabla}$ is a symmetric affine connection, we have

$$T_{kj}{}^h - T_{jk}{}^h = \frac{1}{8}[\psi, \psi]_{kj}{}^h + \frac{1}{8}[\theta, \theta]_{kj}{}^h - \frac{1}{4}[\phi, \psi]_{kj}{}^t \theta_t{}^h = 0$$

by virtue of theorem 6.2.

Thus, we have

THEOREM 7.1. *Suppose that three 1-forms u, v and w are all closed in a normal (ϕ, ψ)-manifold with complemented frames $M(\phi, \psi, U, V, W, u, v, w)$. Then there exists in M a symmetric affine connection ∇ such that $\nabla\phi=0$, $\nabla\psi=0$, $\nabla\theta=0$, $\nabla U=0$, $\nabla V=0$, $\nabla W=0$, $\nabla u=0$, $\nabla v=0$ and $\nabla w=0$.*

Combining theorem 3.2 and 5.3, we have

THEOREM 7.2. *On a normal metric (ϕ, ψ)-manifold with complemented frames M, if one of two structures (ϕ, g, u, v, w), (ψ, g, u, v, w) is cosymplectic, then M admits an almost contact metric 3-structure (F, G, H, g, u, v, w) such that $\nabla F=0$, $\nabla G=0$, $\nabla H=0$, $\nabla U=0$, $\nabla V=0$, $\nabla W=0$, $\nabla u=0$, $\nabla v=0$ and $\nabla w=0$, where ∇ is a certain symmetric affine connection.*

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