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A CHARACTERIZATION OF STRICTLY CONVEX 2-NORMED SPACES

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Let L be a real vector space of dimension greater than 1. A 2 *norm* for L is a real-valued function $\|\ ,\ \|$ on $L\times L$ which satisfies:

1. $\|a,b\|=0$ iff a and b are dependent.
2. $\|a,b\|=\|b,a\|$.
3. $\|\tau a,b\|=|\tau|\,\|a,b\|$.
4. $\|a,b+c\|\leq\|a,b\|+\|a,c\|$.

$(L\ \|\ ,\ \|)$ is called a *2-normed space*. ([2], pg. 2)

If $a_1, a_2, \cdots, a_n$ are elements of L, let $V(a_1, a_2, \cdots, a_n)$ denote the subspace of L generated by $a_1, a_2, \cdots, a_n$.

DEFINITION A. $(L,\ \|\ ,\ \|)$ is *strictly convex* if $\|a,c\|=\|b,c\|=\frac{1}{2}\|a+b,c\|=1$ and $c\notin V(a,b)$ imply that $a=b$.

It is easily shown that the following is an equivalent formulation of definition A.

DEFINITION A′. $(L,\ \|\ ,\ \|)$ is strictly convex if $\|a,c\|=\|b,c\|=1$, $c\notin V(a,b)$, and $a\neq b$ imply that $\|a+b,c\|<2$.

For further properties of strictly convex 2-normed spaces, see [1].

DEFINITION B. If M and N are linear subspaces of L, a bilinear form F on $M\times N$ is said to be *bounded* if there is a number $K>0$ for which $|F(a,b)|\leq K\|a,b\|$ for every $(a,b)\in M\times N$.

The norm of F, $\|F\|$, is defined by:

$\|F\|=\inf K: |F(a,b)|\leq K\|a,b\|$ for every $(a,b)\in M\times N$. Additional information about bounded bilinear forms on 2-normed spaces may be found in [3] and [5].

THEOREM 1. *The following are equivalent*:

1. $(L,\ \|\ ,\ \|)$ *is strictly convex.*
2. *If* $c\neq 0$, F *is a non-zero bounded bilinear form on* $L\times V(c)$, $\|x,c\|=\|y,c\|=1$ *and* $F(x,c)=F(y,c)=\|F\|$, *then either* $x=y$ *or* $\|x,y\|\neq 0$ *and* $c=\pm\frac{1}{\|x,y\|}(x-y)$.

Proof: A. Assume $(L,\ \|\ ,\ \|)$ is strictly convex. Let $c\neq 0$ and F be a non-zero bounded bilinear form on $L\times V(c)$. If $F(x,c)=F(y,c)=\|F\|$ and $\|x,c\|=\|y,c\|=1$, then $2=\frac{1}{\|F\|}F(x+y,c)\leq\|x+y,c\|\leq\|x,c\|+\|y,c\|=2$. Therefore $\|x+y,c\|=2$. If $x\neq y$, then $c\in V(x,y)$ since otherwise the strict convexity of L would yield $x=y$. Hence, there are real numbers α and β for which $c=\alpha x+\beta y$.

Then, $1=\|x,c\|=\|x,\alpha x+\beta y\|=|\beta|\,\|x,y\|$. (see [2], pg. 5) Similarly $|\alpha|\,\|x,y\|=1$. Therefore, $\|x,y\|\neq 0$ and $|\alpha|=|\beta|=\frac{1}{\|x,y\|}$. Since $\|x+y,c\|=2$, it follows that $c=\pm\frac{1}{\|x,y\|}(x-y)$.

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B. Assume condition 2 holds and let $\|a, c\|=\|b, c\|=1$, $a \neq b$, and $c \notin V(a, b)$. Then, $\|a+b, c\| \leq \|a, c\|+\|b, c\|= 2$. If $\|a+b, c\|=2$, then by [5], theorem 2.8, pg. 56, there is a bounded bilinear form F defined on $L \times V(c)$ such that $\|F\|=1$ and $F\left(\frac{a+b}{2}, c\right)=\|\frac{a+b}{2}, c\|=1$.

Note that $F(a, c) \leq |F(a, c)| \leq \|F\| \|a, c\|=1$. If $F(a, c)=1$, then since $a \neq \frac{a+b}{2}$, condition 2 with $x=a$ and $y=\frac{a+b}{2}$ implies that $c \in V(a, b)$ which is impossible. Thus, $F(a, c)<1$. A similar argument shows that $F(b, c)<1$ also. Therefore,

$$1=\frac{1}{2}F(a+b, c)=\frac{1}{2}F(a, c)+\frac{1}{2}F(b, c)<1.$$

Hence, $\|a+b, c\|<2$ and $(L, \| \ , \ \|)$ is strictly convex by definition A′.

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