J. Korean Math. Soc.
Vol. 11, No. 1, 1974

DISJOINT ESSENTIAL ANGULAR CLUSTER SETS

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1. We consider measurable functions f from the upper half plane $H; Im(z)>0$ into an arbitary compact metric space (Ω, d). A Stolz angle at a point x on the real line R is any region of the form

$$\Delta=\{z\varepsilon H:\ p<arg(z-x)<q\}\quad (0<p<q<\pi).$$

For each Stolz angle Δ at $x\in R$, we define the essential angular cluster set $C_e(f,x,\Delta)$ of f at x relative to Δ as follows: the point $a\in\Omega$ is in $C_e(f,x,\Delta)$ if, for every $\varepsilon>0$,

$$\lim_{r\to 0}\sup\frac{m[\Delta(r)\cap f^{-1}(\{b\in\Omega: d(a,b)<\varepsilon\})]}{m\Delta(r)}>0$$

where $\Delta(r)=\Delta\cap\{z: |z-x|<r\}$ $(r>0)$ and m denotes 2-dimensional Lebesgue measure. We note that $C_e(f,x,\Delta)$ is a compact subset of Ω and that $m[\Delta(r)\cap f^{-1}(G)]/m\Delta(r)\to 1$ as $r\to 0$ for any open set G containing $C_e(f,x,\Delta)$.

We let $T(f)$ denote the set of all points $x\in R$ at which there exist two Stolz angles Δ_x^1 and Δ_x^2 such that $C_e(f,x,\Delta_x^1)\neq C_e(f,x,\Delta_x^2)$. A. M. Bruckner and Casper Goffman [2] have shown that $T(f)$ is of first category. Subsequently, Casper Goffman and W. T. Sledd [3] showed that $T(f)$ is of linear measure zero, but could be uncountable. Our purpose is to show that, although $T(f)$ might be uncountable, at only countably many points $x\in T(f)$ can there be found two Stolz angles Δ_x^1 and Δ_x^2 for which

$$C_e(f,x,\Delta_x^1)\cap C_e(f,x,\Delta_x^2)=\phi.$$

LEMMA. *If S is a measurable subset of H, the set $A(S)$ of all points $x\in R$ at which there exist two Stolz angles Δ_x^1 and Δ_x^2 with*

$$\lim_{r\to 0}\frac{m[\Delta_x^1(r)\cap S]}{m\Delta_x^1(r)}=1 \text{ and } \lim_{r\to 0}\frac{m[\Delta_x^2(r)\cap s]}{m\Delta_x^2(r)}=0$$

is countable.

Proof. Fix four numbers $0<\alpha<\beta<\gamma<\delta<\pi$, and set $\Delta_x^1=\{z: \alpha<\arg(z-x)<\beta\}$ and $\Delta_x^2=\{z:\gamma<\arg(z-x)<\delta\}$ for each $x\in R$. For an arbitary pair of points $x<y$ in R we set $Q_{xy}=\Delta_x^1\cap\ \Delta_y^2$ and we define

$$r_{xy}=\sup_{z\in Q_{xy}}|z-x| \text{ and } r_{yx}=\sup_{z\in Q_{xy}}|z-y|.$$

Through elementary calculations we determine positive numbers K_1, K_2, K_3 (each independent of x and y) such that

$$m\Delta_x^1(r_{xy})=K_1(y-x)^2,\ m\Delta_y^2(r_{yx})=K_2(y-x)^2 \text{ and } mQ_{xy}=K_3(y-x)^2.$$

(Note that K_3/K_1 and K_3/K_2 are both positive numbers less than 1.) Choose a number

Received by the editors Jun. 6, 1974.

$r>0$ and let $A_r[S;\alpha,\beta,\gamma,\delta]$ denote the set of all points $x\in R$ for which

$$\frac{m[\Delta_x^1(t)\cap S]}{m\Delta_x^1(t)}>1-(K_3/2K_1) \text{ and } \frac{m[\Delta_x^2(t)\cap S]}{m\Delta_x^2(t)}\leq K_3/2K_2$$

for each $t\epsilon(0,r)$. Pick a number $\tau_r>0$ so that both $r_{xy}<r$ and $r_{yx}<r$ whenever $0<y-x<\tau_r$.

Suppose there exist points $x, y\in A_r[S;\alpha,\beta,\gamma,\delta]$ with $0<y-x<\tau_r$. It is clear that

$$m(Q_{xy}\cap S)\geqq m[\Delta_x^1(r_{xy})\cap S]-m[\Delta_x^1(r_{xy})-Q_{xy}].$$

Since $x\in A_r[S;\alpha,\beta,\gamma,\delta]$ and since

$$\frac{m[\Delta_x^1(r_{xy})-Q_{xy}]}{m\Delta_x^1(r_{xy})}=1-\frac{K_3}{K_1},$$

it follows that

$$\frac{m(Q_{xy}\cap S)}{m\Delta_x^1(r_{xy})}>[1-\frac{K_3}{2K_1}]-[1-\frac{K_3}{K_1}]=\frac{K_3}{2K_1}.$$

In view of this inequality and the identity

$$\frac{m\Delta_x^1(r_{xy})}{m\Delta_y^2(r_{yx})}=\frac{K_1}{K_2},$$

we see that

$$\frac{m[\Delta_y^2(r_{yx})\cap S]}{m\Delta_y^2(r_{yx})}\geqq\frac{m[Q_{xy}\cap S]}{m\Delta_y^2(r_{yx})}=\frac{m[Q_{xy}\cap S]}{m\Delta_x^1(r_{xy})}\cdot\frac{m\Delta_x^1(r_{xy})}{m\Delta_y^2(r_{yx})}>\frac{K_3}{2K_2}$$

which contradicts $y\in A_r[S;\alpha,\beta,\gamma,\delta]$. Therefore, $|x-y|\geq\tau_r$ for each pair $x, y\in A_r$ $[S;\ \alpha,\beta,\gamma,\delta]$; and hence, $A_r[S;\ \alpha,\beta,\gamma,\delta]$ is a countable set.

To complete the proof, we observe that

$$A[S]\subset A_r[S;\ \alpha,\beta,\gamma,\delta]\cup A_r[H-S;\ \alpha,\beta,\gamma,\delta])$$

where the union is taken over all rational numbers $r>0$ and all 4-tuples $(\alpha,\beta,\gamma,\delta)$ of rational numbers satisfying $0<\alpha<\beta<\gamma<\delta<\pi$.

THEOREM. *If $f:H\to\Omega$ is measurable, the set E of all points $x\in R$ at which there exist two Stolz angles Δ_x^1 and Δ_x^2 with*

$$C_e(f,x,\Delta_x^1)\cap C_e(f,x,\Delta_x^2)=\phi$$

is countable.

Proof. Let B be a countable basis for the metric topology on Ω, and let $\mathcal{G}$ denote the collection of all sets expressible as a finite union of sets in B. Using the compactness of Ω, one can easily show that in the notation of our lemma

$$E\subset\bigcup_{G\in\mathcal{G}}A[f^{-1}(G)];$$

and the theorem is proved.

2. Here we use our theorem to obtain a short proof of a well known result concerning Lebesgue density.

A measurable set $Q\subset R$ is said to have *right density α at the point $x\in R$* if

$$\lim_{a\to0+}\frac{m[Q\cap(x,x+a)]}{a}=\alpha,$$

where $\bar{m}$ denotes linear Lebesgue measure. Left density is defined analogously.

COROLLARY. *If $Q \subset R$ is measurable, the set $B[Q]$ of all points $x \in R$ at which Q has right density* 1 *and left density* 0 *is countable.*

Proof. Define the function f in H to be the characteristic function of $Q \times (0, \infty)$, where $\times$ denotes cartesian cross product. At each point $x \in B[Q]$ choose Stolz angles Δ_x^1 and Δ_x^2 such that $\Delta_x^1(\Delta_x^2)$ lies to the right (left) of the vertical at x. Then

$$C_e(f, x, \Delta_x^1) \cap C_e(f, x, \Delta_w^2) = \{1\} \cap \{0\} = \phi$$

for each $x \in B[Q]$; and, according to our theorem, $B[Q]$ is countable.

3. For the sake of completeness, we now show that the set of points $x \in T(f)$ at which there exist three Stolz angles Δ_x^1, Δ_x^2, Δ_x^3 for which

$$C_e(f, x, \Delta_x^1) \cap C_e(f, x, \Delta_x^2) \cap C_e(f, x, \Delta_x^3) = \phi$$

need not be countable.

Let K denote the Cantor "middle half" subset of $[0, 1]$. Let $l_1(x), l_2(x), l_3(x)$ be the half rays eminating from x and making respective angles $\pi/4$, $\pi/2$, $3\pi/4$ with respect to the positive real axis. Set $S_j = \bigcup_{x \in K} l_j(x)$, $j = 1, 2, 3$. As indicated by F. Bagemihl, G. Piranian and G.S. Young [1, proof of Theorem 2, p. 30], we have $S_1 \cup S_2 \cup S_3 = \phi$. Through an elementary but tedious computation, one can show that

$$\rho(S_2, S_1 \cap S_3) = \frac{1}{2} \ ln \ \frac{14 + \sqrt{187}}{9\sqrt{2}}$$

where $\rho(\cdot, \cdot)$ represents the non-Euclidean hyperbolic metric on H:

$$\rho(z_1, z_2) = \frac{1}{2} \ ln \ \frac{|1 - \bar{z}_1 z_2| + |z_1 - z_2|}{|1 - \bar{z}_1 z_2| - |z_1 - z_2|}$$

For each $\varepsilon < 0$, the set

$$\Delta_x^j(\varepsilon) = \{z \in H : \rho(z, l_j(x)) < \varepsilon\} \quad (j = 1, 2, 3)$$

is a Stolz angle at x. Furthermore, Since $\rho(S_2, S_1 \cap S_3) > 0$, it is clear that there exists an $\varepsilon_0 > 0$ such that

$$\bigcap_{j=1}^{3} \left[\bigcup_{x \in K} \Delta_x^j(\varepsilon_0) \right] = \phi.$$

It is now a trivial matter to define a continuous complex valued function f in H such that

$$\bigcap_{j=1}^{3} C_e(f, x, \Delta_x^j(\varepsilon_0)) = \phi.$$

References

[1] F. Bagemihl, G. Piranian and G. S. Young, *Intersections of cluster sets,* Bul. Inst. Politehn. Iasi (N. S.) 5, (1959), 29–34.

[2] A. M. Bruckner and Casper Goffman, *The boundary behavior of real functions in the upper half plane,* Rev. Roumaine de Mat. Pures et Appl., XI (1966), 507–518.

[3] Casper Goffman and W. T. Sledd, *Essential cluster sets,* J. London Math. Soc. (2), 1 (1969), 295–302.

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