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LINEARLY SEMISTRATIFIABLE SPACES

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0. Introduction.

In this paper, a class of spaces, called linearly semistratifiable spaces, is introduced. Linearly stratifiable spaces and semistratifiable spaces are linearly semistratifiable. Many properties of linearly stratifiable spaces and of semistratifiable spaces can be shared with linearly semistratifiable spaces.

In Section 1, characterizations of linearly semistratifiable spaces are given. From these it is proved that a space is semistratifiable if and only if it has a σ-cushioned pair-net. In Section 2, properties of linearly semistratifiable spaces are investigated. The main results are: If X is a space semistratifiable over α, then

A. Every open set in X is an F_α-set,

B. Every subspace of X is semistratifiable over α,

C. X is F_α-screenable and

D. Every closed continuous image of X is semistifiable over α.

It is also proved that a space is semistratifiable over α if and only if it is dominated by a collection of closed subsets, each of which is semistratifiable over α. In Section 3, product theorems are proved. In Section 4, it is shown that in a space which is semistratifiable over α, the following are equivalent: α-Lindelöf property, hereditarily α-separability and the property that every subset with cardinality greater than α has a limit point. In the final section related examples are given.

In this paper, no separation axioms are preassumed. We denote the closure of a subset A of a topological space by $cl\,A$. All undefined terms and symbols are as in [7].

1. Definitions and characterizations.

DEEFINITION 1. 1. An ordinal number α is called an *initial ordinal* number provided for every ordinal $\beta<\alpha$, there exists an injection from β to α, but there does not exist an injection from α to β. We assume that cardinal numbers and initial ordinal numbers are the same. Let ω stand for the first infinite ordinal, and Ω for the first uncountable ordinal.

DEFINITION 1. 2. Let (X, τ) be a topological space and α be an initial ordinal not less than ω. The space X is said to be *semistratifiable over* α or *linearly semistratifiable* provided that there exists a map S: $\alpha\times\tau \longrightarrow$ {closed subsets of X} (called an α-*semistratification*) which satisfies the following

LSS$_1$: For every $U\in\tau$, $U=\bigcup\{S(\beta, U): \beta<\alpha\}$.

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LSS$_2$: If U, $V\in\tau$ and $U\subset V$, then $S(\beta, U)\subset S(\beta, V)$ for all $\beta<\alpha$.

LSS$_3$: If $\gamma<\beta<\alpha$, then $S(\gamma, U)\subset S(\beta, U)$ for all $U\in\tau$.

We will denote $S(\beta, U)$ by U_β unless any confusion occur.

DEFINITION 1.3. A space X is said to be *α-semistratifiable* provided α is the smallest initial ordinal for which X is semistrarifiable over α. A space which is semistratifiable over ω is said to be *semistratifiable*, and the map S is called a *semistratification.*

In the case of semistratifiable space, our definition above agrees with that of Creede [5] because if S is a semistratification which satisfies LSS$_1$ and LSS$_2$, then there is another semistratification which satisfies all three conditions LSS$_1$, LSS$_2$ and LSS$_3$. Example 5.4 shows that this is not true in general for $\alpha>\omega$.

DEFINITION 1.4. ([5], Creede) Let $\mathcal{P}$ be collection of ordered pairs $\boldsymbol{P}=(P_1, P_2)$ of subsets of X with $P_1\subset P_2$ for all $\boldsymbol{P}\in\mathcal{P}$. $\mathcal{P}$ is said to be *cushioned* if for every $\mathcal{P}'\subset\mathcal{P}$,

$$cl(\bigcup\{P_1:\ \boldsymbol{P}\in\mathcal{P}'\})\subset\bigcup\{P_2:\ \boldsymbol{P}\in\mathcal{P}'\}.$$

$\mathcal{P}$ is *σ-cushioned* if it is a union of countably many cushioned subcollections.

DEFINITION 1.5. ([11], Vaughan) A collection $\mathcal{P}$ of pairs $\boldsymbol{P}=(P_1, P_2)$ of subsets of a space X is said to be *linearly cushioned collection of pairs with respect to a linear order* $\leqq$ provided $\leqq$ is a linear order on $\mathcal{P}$ such that

$$cl(\bigcup\{P_1:\ \boldsymbol{P}=(P_1,P_2)\in\mathcal{P}'\})\subset\bigcup\{P_2:\ \boldsymbol{P}=(P_1, P_2)\in\mathcal{P}'\}$$

for every subset $\mathcal{P}'$ of $\mathcal{P}$ which is majorized with respect to $\leqq$.

DEFINITION 1.6. A collection $\mathcal{P}$ of pairs $\boldsymbol{P}=(P_1, P_2)$ of subsets of a space (X,τ) is called a *pair-net* provided that for every x in X and every open U containing x, there exists a $\boldsymbol{P}=(P_1, P_2)\in\mathcal{P}$ such that $x\in P_1\subset P_2\subset U$.

THEOREM 1.7. *If (X,τ) is a space and α an infinite initial ordinal, then the following are equivalent:*

(1) *X is semistratifiable over α.*

(2) *X has a linearly cushioned pair-net $\mathcal{P}$ with which α is confinal.*

(3) *There is a function g from $\alpha\times X$ into τ such that* (a) *for each $x\in X$, $x\in\bigcap\{g(\beta, x):\beta<\alpha\}$;* (b) *if $x\in g(\beta, x_\beta)$ for each $\beta<\alpha$, then the net $\{x_\beta:\ \beta<\alpha\}$ accumulates at x; and* (c) *if $\gamma<\beta<\alpha$, then $g(\gamma, x)\supset g(\beta, x)$ for every $x\in X$.*

(4) *There is a function g from $\alpha\times X$ into τ such that* (a) *for each $x\in X$, $\bigcap\{g(\beta, x):\ \beta<\alpha\}=cl\{x\}$;* (b) *if $x\in g(\beta, x_\beta)$ for each $\beta<\alpha$, then the net $\{x_\beta:\ \beta<\alpha\}$ converges to x; and* (c) *if $\gamma<\beta<\alpha$, then $g(\gamma, x)\supset g(\beta, x)$ for every $x\in X$.*

Proof. (1) $\Longrightarrow$ (2). Let S be an α-semistratification for X. Let any well-order be given on τ. Define

$$\mathcal{P}=\{(U_\beta, U):\ (\beta, U)\in\alpha\underset{lex}{\times}\tau\}$$

where $\alpha\underset{lex}{\times}\tau$ denotes the product set $\alpha\times\tau$ with the lexicographic order.

For any $U\in\tau$ containing x, there exists a $\beta<\alpha$ such that x belongs to U_β by LSS$_1$. Thus, $x\in U_\beta\subset U=U$. This shows that $\mathcal{P}$ is a pair-net for (X,τ).

To show that $\mathcal{P}$ is linearly cushioned, let $\mathcal{P}'$ be a subclass of $\mathcal{P}$ with an upper bound (U_β, U). For any V such that $(V_\gamma, V)\in\mathcal{P}'$ for some $\gamma<\alpha$, V_β is contained in the closed set $(\bigcup\{W: (W_\gamma, W)\in\mathcal{P}' \text{ for some } \gamma\})_\beta$. Thus,

$$cl(\bigcup\{V_\beta: (V_\gamma, V)\in\mathcal{P}')\subset(\bigcup\{V: (V_\gamma, V)\in\mathcal{P}'\})_\beta.$$

Consequently it follows that

$$\begin{aligned} cl(\bigcup\{V_\gamma: (V_\gamma, V)\in\mathcal{P}')&\subset cl(\bigcup\{V_\beta: (V_\gamma, V)\in\mathcal{P}') \\ &\subset(\bigcup\{V:(V_\gamma, V)\in\mathcal{P}'\})_\beta \\ &\subset\bigcup\{V:(V_\gamma, V)\in\mathcal{P}'\}. \end{aligned}$$

Since α is an infinite initial ordinal, α is clearly cofinal with $\mathcal{P}$. This completes the proof.

(2) $\Longrightarrow$ (3). Let $\mathcal{P}$ be a linearly cushioned pair-net for X, and α confinal with $\mathcal{P}$. There is a subclass $\mathcal{P}'=\{\boldsymbol{P}_\beta: \beta<\alpha\}$ such that for every $\boldsymbol{P}\in\mathcal{P}$ there is a $\beta<\alpha$ such that $\boldsymbol{P}\leqq\boldsymbol{P}_\beta$.

For each x in X and each $\beta<\alpha$, define

$$g(\beta, x)=X-cl(\bigcup\{P_1: x\notin P_2 \text{ and } \boldsymbol{P}=(P_1, P_2)\leqq\boldsymbol{P}_\beta\}).$$

Clearly (a) and (c) hold. To show (b) holds, assume x is not a cluster point of a net $\{x_\beta: \beta<\alpha\}$. There is a neighborhood V of x such that $\{x_\beta: \beta<\alpha\}$ is not frequently in V. Since α is well-ordered, there is a $\gamma_0<\alpha$ such that $x_\gamma\notin V$ for any $\gamma\geqq\gamma_0$. So $x\notin cl(\{x_\gamma:\gamma_0\leqq\gamma<\alpha\})$. This means that there is a $\boldsymbol{P}=(P_1, P_2)$ in $\mathcal{P}$ such that $x\in P_1 \subset P_2\subset X-cl(\{x_\gamma:\gamma_0\leqq\gamma<\alpha\})$. Let $\beta<\alpha$ be such that $\boldsymbol{P}\leqq\boldsymbol{P}_\beta\in\mathcal{P}'$. Then

$$x\in cl(\bigcup\{P_1: x_\gamma\notin P_2 \text{ and } \boldsymbol{P}\leqq\boldsymbol{P}_\beta\})$$

for all $\gamma\geqq\gamma_0$. That is,

$$x\notin g(\beta, x_\gamma) \text{ for all } \gamma\geqq\gamma_0.$$

Take $\delta=\max\{\beta, \gamma_0\}$. Then $x\notin g(\delta, x_\delta)$.

(3) $\Longrightarrow$ (4). Let g be a map as is described in (3). (a) Let $y\in\bigcap\{g(\beta, x): \beta<\alpha\}$. Then $y\in\bigcap\{g(\beta, x_\beta): \beta<\alpha\}$ with $x_\beta=x$ for all $\beta<\alpha$. By (b), y is a cluster point of the net $\{x, x, x, \cdots\}$. This means $y\in cl\{x\}$.

Conversely, assume $y\in cl\{x\}$. For each $\beta<\alpha$, $g(\beta, y)\cap\{x\}\neq\phi$ since $g(\beta, y)$ is a neighborhood of y. This implies that $x\in\bigcap\{g(\beta, y);\ \beta<\alpha\}$. As in the above proof, $y\in cl\{x\}$. Again this means that $g(\beta, x)\cap\{y\}\neq\phi$ for all $\beta<\alpha$, which is equivalent to $y\in\bigcap\{g(\beta, x): \beta<\alpha\}$.

(b) Let $x\in g(\beta, x_\beta)$ for all $\beta<\alpha$ and assume that $\{x_\beta:\beta<\alpha\}$ does not converge to x. Since $\{x_\beta: \beta<\alpha\}$ accumulates at x, there exist a neighborhood W of x and a cofinal subset D of α such that $\{x_\gamma:\gamma\in D\}\cap W=\phi$. Define a new net as follows:
If $\beta\in D$, then let $y_\beta=x_\beta$. If $\beta\in\alpha-D$, let $y_\beta=x_\gamma$, where γ is the first element of D which follows β.

Then $x\in g(\beta, y_\beta)$ for all $\beta<\alpha$ since $g(\beta, y_\beta)$ is either $g(\beta, x_\beta)$ or $g(\beta, x_\gamma)$ with $\beta<\gamma$. But $g(\beta, x_\gamma)\supset g(\gamma, x_\gamma)$; in either case, $g(\beta, y_\beta)$ contains x. Thus, x is a cluster point of $\{y_\beta: \beta<\alpha\}$, and hence is a cluster point of $\{x_\gamma: \gamma\in D\}$, which is a contradiction. This completes the proof.

(4) $\Longrightarrow$ (1). Let g be a map as is described in (4). Define a map $S: \alpha\times\tau\longrightarrow$ {closed subsets of X} by

$$S(\beta, U)=X-\bigcup\{g(\beta, x): x\in X-U\}.$$

$S(\beta, U)\subset X-(X-U)=U$ since $x\in g(\beta, x)$ for all $\beta<\alpha$.
Conversely, assume $x\notin \bigcup\{S(\beta, U): \beta<\alpha\}$. Then $x\in \bigcup\{g(\beta, y): y\in X-U\}$ for all $\beta<\alpha$. This implies there is an $y_\beta\in X-U$ such that $x\in g(\beta, y_\beta)$ for each $\beta<\alpha$. Thus $\{y_\beta:\beta<\alpha\}$ satisfies the condition (b) of (4), and hence converges to x. Since $X-U$ is closed, we have $x\in cl(\{y_\beta: \beta<\alpha\})\subset X-U$. Thus the proof is completed.

The following corollary is another characterization of semistratifiable spaces by means of σ-cushioned pair-net.

COROLLARY 1.8. *A necessary and sufficient condition for a space to be semistratifiable is that the space has a σ-cushioned pair-net.*

Proof. By the remark following the definition 1.3, X is semistratifiable if and only if it is semistratifiable over ω. The theorem 1.7 shows that X has a linearly cushioned pair-net $\mathcal{P}$ with which ω is cofinal. Let i be the embedding of ω into $\mathcal{P}$ and $i(n)=\boldsymbol{P}_n$. Let

$$\mathcal{P}_n=\{\boldsymbol{P}\in\mathcal{P}:\boldsymbol{P}_n\leq \boldsymbol{P}<\boldsymbol{P}_{n+1}\}$$

for each positive integer n. Since each $\mathcal{P}_n$ is majorized by $\boldsymbol{P}_{n+1}$, $\mathcal{P}_n$ is cushioned.
Conversely, it is easily verified that a σ-cushioned collection is linearly cushioned. This completes the proof.

Linearly semistratifiability is a generalization of both semistratifiability and linearly stratifiability ([11]). As the example 5.2 shows, there exist linearly semistratifiable spaces which fail to be semistratifiable or linearly stratifiable.
The next characterization justifies the terminology *linearly* semistratifiable.

PROPOSITION 1.9. *Let (X,τ) be a space. X is linearly semistratifiable iff there is a linearly ordered set A and a map S: $A\times\tau \longrightarrow$ {closed subsets of X} which satisfies LSS_1, LSS_2 and LSS_3.*
We omit the proof (see [11], Proposition 2.9). As in the case of linearly stratifiable spaces, if X is α-semistratifiable, then α is a regular (i.e., there exists no strictly smaller ordinal which is cofinal with α) initial ordinal.
The following is a generalization of Vaughan's result, which is useful to make examples.

PROPOSITION 1.10. *If (X,τ) is a T_1-space which is semistratifiable over a regular infinite initial ordinal α, then every subset F whose cardinality is strictly less than α is a closed discrete subspace.*

2. Properties of linearly semistratifiable spaces.

We shall now give some results for linearly semistratifiable spaces which can be easily extended from the analogous results for semistratifiable spaces and linearly stratifiable spaces.

DEFINITION 2.1. Let m be an infinite cardinal number. A subset of a space X is called an F_m-set if it is a union of a collection $\mathcal{C}$ of closed subsets with the cardinality of $\mathcal{C}$ less than or equal to m. An $F_{\aleph_0}$-set is called an F_σ-set as usual.

THEOREM 2.2. *Every open set in a space which is semistratifiable over α is an F_α-set.*

THEOREM 2.3. *Any subspace of a space which is semistratifiable over α is semistratifiable over α.*

Proof. Let S be an α-semistratification of X, and Y be a subspace of X. Define

$$S': \alpha\times\tau_Y \longrightarrow \{\text{closed subsets of } Y\}$$

by the restriction of S to τ_Y-open subsets of X. It is easily verified that S' is an α-semistratification for Y.

DEFINITION 2.4. Let α be an infinite initial ordinal. A space X is said to be *F_α-screenable* iff every open cover of X has an α-discrete closed refinement. An F_ω-screenable space is said to be *F_σ-screenable* (or *subparacompact*, [3]).

J. G. Ceder [4] proved that stratifiable spaces are paracompact, and J. E. Vaughan [11] generalized this proposition to linearly stratifiable spaces. On the other hand, G. D. Creede [5] showed that semistratifiable spaces are subparacompact. The present auther do not know whether linearly semistratifiable spaces are subparacompact or not. We prove only the following

THEOREM 2.5. *A space which is semistratifiable over α is F_α-screenable (hence, is hereditarsly F_α-screenable).*

Proof. Let S be an α-semistratification for X and let an open cover $\mathcal{U}$ of X be given. Give an well-order $\leqq$ on $\mathcal{U}=\{U_t:t\in I\}$. We will construct an α-discrete closed refinement $\mathcal{H}$ by transfinite construction. For any $\beta<\alpha$, define

$$H_{1\beta}=S(\beta, U_1),$$
$$H_{t\beta}=S(\beta, U_t)-\bigcup\{U_s:\ s\in I \text{ and } s<t\}$$

and let $\mathcal{H}_\beta=\{H_{t\beta}:t\in I\}$.

To show that each $\mathcal{H}_\beta$ is discrete, let $V_1=U_1$ and

$$V_t=U_t-S(\beta, \bigcup\{U_s:\ s<t\}).$$

Then $H_{t\beta}\subset V_t$ for each $t\in I$. Moreover, each V_t meets exactly one member of $\mathcal{H}_\beta$, say $H_{t\beta}$.

For any x in X, let t be the first element of I such that $x\in U_t$. Since $U_t=\bigcup\{S(\beta, U_t):\ \beta<\alpha\}$, there is a $\beta<\alpha$ such that $x\in S(\beta, U_t)$. Then $x\in H_{t\beta}$. That is, $\mathcal{H}=\bigcup\{\mathcal{H}_\beta:\ \beta<\alpha\}$ covers X.

THEOREM 2.6. *Closed continuous image of a space which is semistratifiable over α is semistratifiable over α.*

Proof. Let f be a closed continuous surjection from X to Y and let S be an α-semistratification for X. Define

$$T: \alpha\times\tau_Y \longrightarrow \{\text{closed subsets of } Y\}$$

by $T(\beta, V)=f(S(\beta, f^{-1}(V)))$. Then T is an α-semistratification for Y.

LEMMA 2.7. *Let X be semistratifiable over α and Y be a closed subspace of X with an α-semistratification S. Then there is an α-semistratification T for X such that*

$$S(\beta, V\cap Y)=T(\beta, V)\cap Y$$

for every $\beta < \alpha$ *and every open* (*in* X) V.

Proof. Let S' be any α-semistratification for X. Define an α-semistratification T for X as follows:

$$T(\beta, V) = S(\beta, V \cap Y) \cup S'(\beta, V - Y).$$

Now it is clear that T satisfies all the desired conditions.

LEMMA 2.8. *The union of two closed* (*in the union*) *subspaces which are semistratifiable over* α *is also semistratifiable over* α.

Proof. Apply Lemma 2.7 with respect to the common subspace.

COROLLARY 2.9. (Creede) *The union of two closed* (*in the union*) *semistratifiable spaces is semistratifiable*.

DEFINITION 2.10. (Michael) Let X be a space and $\mathscr{B}$ a collection of closed subsets of X. Then $\mathscr{B}$ *dominates* X if, whenever $A \subset X$ has a closed intersection with every element of some subcollection $\mathscr{B}_1$ of $\mathscr{B}$ which covers A, A is closed.

THEOREM 2.11. *A space is semistratifiable over* α *iff it is dominated by a collection of closed subsets, each of which is semistratifiable over* α.

Proof. Let $\mathscr{B}$ be a dominating collection of subsets of X, each of which is semistratifiable over α. Consider the class G of all pairs of the form $(\mathscr{C}_t, S_t)$, where $\mathscr{C}_t \subset \mathscr{B}$ and S_t is an α-semistratification for $C_t = \cup \{B\colon B \in \mathscr{C}_t\}$ which will be denoted by $S_t(\beta, V) = V_{t\beta}$ (V relatively open in C_t). We partially order G by letting $(\mathscr{C}_s, S_s) \leqq (\mathscr{C}_t, S_t)$ whenever $\mathscr{C}_s \subset \mathscr{C}_t$ and, for each relatively open U in C_t, $U_{t\beta} \cap C_s = (U \cap C_s)_{s\beta}$ for all $\beta < \alpha$.

We now show that any simply ordered subfamily $\{\mathscr{C}_s, S_s)\colon s \in A\}$ of G has an upper bound $\mathscr{C}_t = \cup \{\mathscr{C}_s\colon s \in A\}$. For each relatively open U in C_t and for all $\beta < \alpha$, let

$$S_t(\beta, U) = U_{t\beta} = \bigcup_{s \in A} (U \cap C_s)_{s\beta}.$$

For each $t' \in A$,

$$\begin{aligned} U_{t\beta} \cap C_{t'} &= \bigcup_{s \in A} (U \cap C_s)_{s\beta} \cap C_{t'} \\ &= \bigcup_{s \in A} \{(U \cap C_s)_{s\beta} \cap C_{t'}\} = (U \cap C_{t'})_{t'\beta}. \end{aligned}$$

Thus, $U_{t\beta} \cap C_{t'}$ is closed in $C_{t'}$ for every $t' \in A$, which implies that $U_{t\beta}$ is closed in X since $\mathscr{B}$ is a dominating collection. Moreover, S_t is an α-semistratification for C_t. Consequently, $(\mathscr{C}_t, S_t)$ is an upper bound of $\{\mathscr{C}_s, S_s\colon s \in A\}$.

Let $(\mathscr{C}_0, S_0)$ be a maximal element of G which exists by Zorn's Lemma. To complete the proof we need only to show that $\mathscr{C}_0 = \mathscr{B}$. Suppose not. Then there exists an $E \in \mathscr{B} - \mathscr{C}_0$. Let $\mathscr{C}_1 = \mathscr{C}_0 \cup \{E\}$. Now C_0 and E are closed subspaces semistratifiable over α, and hence C_1 is semistratifiable over α by Lemma 2.8. Thus, by Lemma 2.7, one may obtain an α-semistratification S_1 of C_1 such that

$$U_{1\beta} \cap C_0 = (U \cap C_0)_{0\beta}$$

for all $\beta < \alpha$. Consequently $(\mathscr{C}_0, S_0) < (\mathscr{C}_1, S_1)$, contradicting the maximality of $(\mathscr{C}_0, S_0)$. Hence $\mathscr{C}_0 = \mathscr{B}$ and X is semistratifiable over α.

The following is a generalization of Creede's result.

COROLLARY 2.12. *A space is semistratifiable iff it is dominated by a collection of semistratifiable subsets.*

3. Products.

In [5] G. D. Creede proved that a countable product of semistratifiable spaces is semistratifiable. In this section, we shall prove that a finite product of spaces semistratifiable over the same α is again semistratifiable over α. But Example 5.3 shows that if $\alpha > \omega$, then a countable product of spaces semistratifiable over α need not be linearly semistratifiable.

LEMMA 3.1. (Vaughan) *Let α be an infinite initial ordinal number, and let $\{A_\lambda: \lambda \in \Lambda\}$ be a family of linearly ordered sets such that α has cardinality greater than that of Λ, and α is cofinal with A_λ for all $\lambda \in \Lambda$. If Λ is finite or α is a regular ordinal, then $A = \Pi\{A_\lambda: \lambda \in \Lambda\}$ can be well-ordered so that for every majorized $H \subset A$ we have $Pr_\lambda(H)$ is majorized in A_λ for all $\lambda \in \Lambda$, and α is cofinal in A. Further, if α is the smallest initial ordinal cofinal with each A_λ, then α is the smallest initial ordinal cofinal with A.*

Modifying the proof of theorem 5.2 of [11] we get the following

THEOREM 3.2. *Let α be an initial ordinal number $\geqq \omega$. Let X_i be semistratifiable over α for each $i < \omega$. Then the following hold;*

A. *$\Pi\{X_i: i \leqq n\}$ is semistratifiable over α for all $n < \omega$.*

B. *If each X_i is α-semistratifiable, then $\Pi\{X_i: i \leqq n\}$ is α-semistratifiable for each $n < \omega$.*

C. (Creede) *If each X_i is semistratifiable, then $\Pi\{X_i: i < \omega\}$ is semistratifiable.*

D. *If each X_λ is semistratifiable over the regular initial ordinal α for all $\lambda \in \Lambda$ and α is strictly larger than the cardinality of Λ, then $\Pi\{X_\lambda: \lambda \in \Lambda\}$ with the box topology ([7], 3V) is semistratifiable over α.*

4. Generalizations of Lindelöf property, separability and $\aleph_1$-compactness.

DEFINITION 4.1. ([8] Lutzer and Bennett, [5] Creede) Let m denote an infinite cardinal number. A space X is *m-separable* if X contains a dense subset having cardinality $\leqq m$; X is *m-Lindelöf* if every open cover of X has a subcover with cardinality $\leqq m$. A space is *$\aleph_1$-compact* if every uncountable subset has a limit point.

LEMMA 4.2. *Let α be an infinite initial ordinal ($=$infinite cardinal), and X be a space in which every open set is an F_α-set* (see 2.1). *If X is α-Lindelöf, X is hereditarily α-Lindelöf.*

Proof. Let Y be a subspace of X, and let an open cover $\mathcal{U}$ of Y be given. Since $V = \bigcup\{U: U \in \mathcal{U}\}$ is open in X, V is an F_α-set. Hence there exist closed sets $\{A_\beta: \beta < \alpha\}$ such that $U = \bigcup\{A_\beta: \beta < \alpha\}$.

For each $\beta < \alpha$, $\{X - A_\beta\} \cup \mathcal{U}$ is an open cover of X. By α-Lindelöf property of X, there is a subfamily $\mathcal{U}_\beta$ of $\mathcal{U}$ with cardinality $\leqq \alpha$ such that $\{X - A_\beta\} \cup \mathcal{U}_\beta$ covers X. Let $\mathcal{V} = \bigcup \mathcal{U}_\beta$. Then $\mathcal{V}$ is a subfamily of $\mathcal{U}$ with cardinality $\leqq \alpha$ which covers Y.

THEOREM 4.3. *In a T_1-space which is semistratifiable over α, the followiag are equivalent;*

(1) *X is α-Lindelöf.*

(2) *X is hereditarily α-separable.*

(3) *Every subset of X with cardinality $>\alpha$ has a limit point.*

Proof. (1) $\Longrightarrow$ (2). By Theorem 2.2 and Lemma 4.2, X is hereditarily α-Lindelöf. It suffices to show that X is α-separable. Let $g:\alpha\times X \longrightarrow \tau$ be a function satisfying the conditions of Theorem 1.6. (4). For each $\beta<\alpha$, $\{g(\beta, x):x\in X\}$ is an open cover of X and, since X is α-Lindelöf, there is a subset D_β of X with cardinality $\leqq\alpha$ such that $\{g(\beta, x): x\in D_\beta\}$ covers X. The set $D=\bigcup\{D_\beta: \beta<\alpha\}$ has cardinality $\leqq\alpha$. To show that D is dense in X, let x be any point of X. For each $\beta<\alpha$, there exists an $x_\beta\in D_\beta$ such that $x\in g(\beta, x_\beta)$. By the condition on g, the net $\{x_\beta: \beta<\alpha\}$ converges to x. Hence, $x\in cl\, D$.

(2) $\Longrightarrow$ (3). Assume there is a subset F with cardinality $>\alpha$ which has no limit point. Each x in F has an open neighborhood U_x which meets F at exactly one point x. This implies that F is dicrete (hence, is not α-sparable) subset of X.

(3) $\Longrightarrow$ (1). Let $\mathcal{U}$ be an open cover of X and suppose that $\mathcal{U}$ has no subcover with cardinality $\leqq\alpha$. By Theorem 2.5, $\mathcal{U}$ has an α-discrete closed refinement $\mathcal{F}=\bigcup\{\mathcal{F}_\beta: \beta<\alpha\}$. Since $\mathcal{U}$ has no subcover with cardinality $\leqq\alpha$, there is a $\gamma<\alpha$ such that $\mathcal{F}_\gamma$ has cardinality $>\alpha$. Let X' be a subset of X consisting of exactly one point of each nonempty element of $\mathcal{F}_\gamma$. The set X' has cardinality $>\alpha$ and has no limit point, since X$'$ is discrete and X is a T_1-space.

When $\alpha=\omega$, we get the following

COROLLARY 4.4 (Creede) *In a semistratifiable T_1-space X, the following are equivalent;*

(1) *X is Lindelöf*,

(2) *X is hereditarily separable* and

(3) *X is $\aleph_1$-compact.*

5. Examples.

EXAMPLE 5.1. There is a compact Hausdorff space which is not linearly semistratifiable. Let $X=[0, \Omega]$ and give a topology as follows: Each point is isolated except Ω, and the basic neighborhood of Ω are co-finite subsets. Since any infinite set missing Ω is not closed, X is not linearly semistratifiable by Proposition 1.10.

EXAMPLE 5.2. There exists a linearly semistratifiable space which is neither semistratifiable nor linearly stratifiable.

Let X be the space of example H described by Bing in [1]. This space is not linearly stratifiable since it is not paracompact, but it is linearly semistratifiable (in fact, it has a σ-discrete net-work). Let $Y=[0, \Omega]$ and equip Y with the smallest topology larger than the order topology for which every point is isolated except Ω. Then Y is an Ω-stratifiable space, which is not semistratifiable since Ω is not a G_δ.

Now let Z be the topological sum of X and Y. Then Z is linearly semistratifiable by Lemma 2.8. Z is neither linearly stratifiable (see [11], Theorem 4.1. B) nor semistratifiable (see [5], Theorem 2.2).

EXAMPLE 5.3. A countable product of Ω-stratifiable spaces need not be linearly semistratifiable. Let X_i be the space Y in Example 5.2 for each $i<\omega$. Since each X_i has isolated points, $X=\Pi\{X_i: i<\omega\}$ has convergent sequences, hence is not linearly semistratifiable by Proposition 1.10.

EXAMPLE 5.4. Every T_1-space (X,τ) has a "semistratification map" $S:\alpha\times\tau\longrightarrow$ {closed subsets of X} which satisfies LSS_1 and LSS_2 of Definition 1.2. Take α to be the cardinal number of X. Well-order X so that $X=\{x_\beta: \beta<\alpha\}$ and define

$$S(\beta,U)=\begin{cases}\{x_\beta\}, & \text{if } x_\beta\in U,\\ \phi, & \text{otherwise.}\end{cases}$$

It is easy to see that S satisfies LSS_1 and LSS_2. Now if this map S also satisfies LSS_3, then X would be linearly semistratifiable. This is impossible (see Example 5.1.).

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