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GENERALIZATION OF DEVELOPABLE SPACES

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0. Introduction

In recent years there have been several generalizations of developable spaces. For example, Bennett [3] defined a quasi-developable space which was proved to be useful in obtaining metrization theorems for M-spaces and linearly orderable topological spaces. Alexander [1] introduced semi-developable spaces and proved that a space is semi-metrizable if and only if it is a semi-developable T_0-space. Also he introduced cushioned pair-semidevelopable spaces [2] to obtain a generalization of Morita's metrization theorem.

In fact, a development for a space has three conditions as shown in the definition 1.1. If we pick out one essential condition, we get a natural generalization of (quasi-/semi-) developments, which we call qs-developments, This paper concerns with qs-developable spaces and with cushioned pair qs-developable spaces. The main results are:

1. A space is semi-developable if and only if it is qs-developable and perfect.
2. Every separable regular T_0-space with a point-finite qs-development is metrizable.
3. A cushioned pair qs-developable T_0-space is a Nagata space.
4. The Sorgenfrey line is not qs-developable.
5. There exists a non-developable (and hence, non-metrizable) cushioned pair qs-developable M_1-space.

Nearly all topological terminologies and symbols appearing in this paper is consistent with that used in Kelly [13]. Exceptions on symbols are closure and interior of a set A, we denote them by $cl(A)$ and $Int(A)$, respectivey. We also adopt the convention that if γ is a collection of sets, then γ^* denotes the union of all sets in γ, and $st(x, \gamma)$ denotes the union of all sets in γ containing x.

1. Quasi-semi-developable spaces.

Let $\gamma=(\gamma_1, \gamma_2, \cdots)$ be a sequence of collections of subsets of a space (X, τ). Consider the the following conditions on γ:

(a) for each $x \in X$, $\{st(x, \gamma_n): n \in Z^+ \text{ and } x \in \gamma_n^*\}$ is a local base at x,

(b) each γ_n is a covering of X and

(c) each γ_n is a subclass of τ.

The above condition (a) is equivalent to the followings:

1) For each $x \in X$ and for each positive integer n such that $st(x, \gamma_n) \neq \phi$, $st(x, \gamma_n)$ is a neighborhood of x, and

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2) For each $x \in X$ and for each open U containing x, there exists a positive integer n such that $x \in st(x, \gamma_n) \subset U$.

DEFINITION 1. 1. γ is called a quasi-semi-development (for brevity, a qs-development) for X if γ satisfies the condition (a). A space is said to be quasi-semi-developable (qs-developable) if it has a qs-development.

Recall that γ is a semi-development [1] if it satisfies (a) and (b); a quasi-development [3] if it satisfies (a) and (c); a development [16] if it satisfies (a), (b) and (c). From the definition it is clear that (semi/quasi-)developable spaces are qs-developable, and that qs-developable spaces are 1-st countable. As the examples 3. 1 and 3. 2 show, no converse is true.

As the case of quasi-developable spaces, the following is true for qs-developable spaces:

THEOREM 1. 2. *A space is semi-developable if and only if it is qs-developable and perfect* (i. e., *every closed set is a* G_δ).

Proof. Let $\gamma = (\gamma_1, \gamma_2, \cdots)$ be a qs-development for a space X. For each $x \in \gamma_i^*$, $st(x, \gamma_i)$ is a neighborhood of x. This implies that each γ_i^* is open. Since X is perfect, each γ_i^* is a F_σ-set. Let $\gamma_i^* = \bigcup_j F_{ij}$, where each F_{ij} is closed. For each i and j, let $\delta_{ij} = \gamma_i \cup \{X - F_{ij}\}$. Now we show that $\{\delta_{ij}\}$ is a desired semi-development.

$st(x, \delta_{ij})$ is a neighborhood of x whenever it is non-empty. Now let R be a neighborhood of x. There is an i such that $x \in st(x, \gamma_i) \subset R$. $\gamma_i^* = \bigcup_j F_{ij}$ implies that there is a j such that x is contained in F_{ij}. It follows that $st(x, \delta_{ij}) = st(x, \gamma_i)$, and hence, $st\ (x, \delta_{ij})$ is a neighborhood of x contained in R.

For the converse, note that semi-developable spaces are perfect.

DEFINITION 1. 3 [10]. A space is said to be z_1-compact if every uncountable subset has a limit point.

DEFINITION 1. 4. A qs-development $\gamma = (\gamma_1, \gamma_2, \cdots)$ is said to be point-finite (point-countable) if each γ_n is point-finite (point-countable).

It is generally known that Lindelöf Moore spaces and $\aleph_1$-compact Moore spaces are metrizable. Bennett [3] proved that in a quasi-developable space, hereditary $\aleph_1$-compactness, hereditary Lindelöf and hereditary separability are equivalent; and each of these conditions implies the metrizability of the space if it is regular. On the other hand, Alexander [1] showed that a separable regular T_0-space with a point-finite semi-development is metrizable. In the case of qs-developable spaces, some modifications are needed.

LEMMA 1. 5. *In a hereditarily* $\aleph_1$*-compact qs-developable space every uncountable subset contains a condensation point.*

THEOREM 1. 6. *In a qs-developable space each of the followings are equivalent:*

1) *hereditarily* $\aleph_1$*-compact,*
2) *hereditarily Lindelöf and*
3) *hereditarily separable.*

Further, if the space is regular T_0 and has a point-finite qs-development, then each of these conditions insures the metrizability of the space.

The first assertion of the above theorem 1.6 is proved using a generalization of the proof for quasi-developable spaces ([3], Theorem 2.5). The remaining part of the theorem 1.6 is an easy consequence of the theorem 1.8.

REMARK. In regular qs-developable T_0-spaces the above conditions 1), 2) and 3) of theorem 1.6 are not sufficient to insure the metrizability of the spaces as seen in [**14**]: Example 3.2 exhibits a paracompact hereditarily separable semi-metric space which is not quasi-developable. Such a space is non-metrizable. This property also distinguishes qs-developable spaces from quasi-developable spaces.

McAuley [**4**] gave an example of separable semi-metric space which is not hereditarily separable, and Alexander [**1**] proved that a separable T_0-space with a point-finite semi-development is hereditarily separable. We generalize these to qs-developable spaces. The following lemma can be proved by analogous method to the proposition 1.12 of [**1**].

LEMMA 1.7. *A separable space with a point-countable qs-development is hereditarily separable.*

THEOREM 1.8. *A separable regular T_0-space with a point-finite qs-development has a point-finite semi-development, and hence is metrizable.*

Proof. Such a space is hereditarily separable by the lemma 1.7, and hence is hereditarily Lindelöf by the theorem 1.6. It is not difficult to show that a regular hereditarily Lindelöf T_0-space is perfect. In the proof of the theorem 1.2, perfect space with a point-finite semi-development has a point-finite semi-development. Use the theorem 1.7 of [**1**].

In [**17**], the notions of θ-base and θ-refinability were introduced in order to characterize developable spaces. Recently, Bennett gave an excellent characterization of quasi-developable spaces by means of θ-base: A space is quasi-developable if and only if it has a θ-base ([**4**], Theorem 8). He also introduced the concept of weak θ-refinability and showed that weak θ-refinability is a sufficient condition for a perfect space to be subparacompact ([**7**]).

The proposition 7 of [**4**] can be generalized as follows:

PROPOSITION 1.9. *A qs-developable space is (hereditarily) weakly θ-refinable.*

COROLLARY 1.10. (McAuley) *A semimetric space is θ-refinable, and hence subparacompact.*

Proof. A semi-metric space is qs-developable and perfect. By the proposition 1.9, it is weakly θ-refinable. Use the theorem 5 of [**4**].

2. Cushioned pair qs-developable spaces.

We now introduce a new class of spaces, cushioned pair qs-developable spaces, which is a generalization of cushioned pair-semidevelopable spaces.

DEFINITION 2.1. If γ and δ are collections of subsets of X, then we say that γ is cushioned in δ if one can assign to each $G\in\gamma$ a $D(G)\in\delta$ such that, for every $\gamma'\subset\gamma$,

$$cl(\cup\{G\colon G\in\gamma'\})\subset\cup\{D(G)\colon G\in\gamma'\}.$$

By a cushioned pair semi-development (qs-development) we shall mean a pair of semi-development (qs-development) (γ, δ) such that γ_n is cushioned in δ_n for each n, and such that

$$(*) \qquad \gamma_1{}^* \subset \gamma_2{}^* \subset \gamma_3{}^* \subset \cdots\cdots$$

The above definition of a cushioned pair semi-development is due to Alexander [2]. In the definition of cushioned pair qs-developments, the condition that

$$(*) \qquad \gamma_1{}^* \subset \gamma_2{}^* \subset \gamma_3{}^* \subset \cdots\cdots$$

is essential. Let X be a qs-developable space with a qs-development $\gamma = (\gamma_1, \gamma_2, \cdots)$. If we set

$$\delta_{2n} = \gamma_n, \ \ \delta_{2n-1} = \{\phi\} \text{ and}$$

$$\eta_{2n} = \{X\}, \ \ \eta_{2n-1} = \gamma_n,$$

then (δ, η) is a cushioned pair qs-development *without the condition* $(*)$.

Clearly a cushioned pair semi-development is a cushioned pair qs-development. That a space has a cushioned pair semi-development is, however, a very strong condition. In fact, we have

THEOREM 2.2 (Alexander) *A space is metrizable if and only if it is T_0 and has a cushioned pair semi-development.*

The following theorem and example 3.3 show that cushioned pair qs-developable spaces locate between metrizable spaces and stratifiable spaces ($=M_3$-spaces, see [6]) and they also distinguish cushioned pair qs-developable spaces from cushioned pair semi-developable spaces.

THEOREM 2.3. *A cushioned pair qs-developable T_0-space is a Nagata spaces ($=$a 1st countable space).*

Proof. Let (γ, δ) be a cushioned pair qs-development for a space X. We may assume that the set of all isolated points of X is contained in $\gamma_1{}^*$. Let

$$\mathcal{P}_n = \{(Int\ st(x, \gamma_n),\ st(x, \delta_n)) : x \in \gamma_n{}^*\}$$

for each n. Then each $\mathcal{P}_n$ is a cushioned collection. To show that $\mathcal{P} = \bigcup \mathcal{P}_n$ is a pair-base, let U be a neighborhood of a point x. If x is an isolated point of X, choose an n such that $x \in st(x, \delta_n) \subset U$. Evidently $x \in Int\ st(x, \gamma_n) \subset st(x, \delta_n) \subset U$. When x is not isolated, choose an m such that $x \in \gamma_m{}^*$. $\bigcap \{st(x, \delta_k) : x \in \delta_k{}^* \text{ and } k < m\} \cap U$ is a neighborhood of x with cardinality $\geqslant 2$. Since δ is a qs-development for X, there is an n such that $x \in st(x, \delta_n) \subsetneqq \bigcup \{st(x, \delta_k) : x \in \delta_k{}^* \text{ and } k < m\} \cap U$. For this n, we have $x \in Int\ st(x, \gamma_n) \subset st(x, \delta_n) \subset U$.

It is not so difficult to show that a cushioned pair qs-developable T_0-space is T_1.

REMARK. A qs-developable T_0-space need not be T_1 as the Sierpinski space shows: $X = \{a, b\}$ with the topology ϕ, $\{x\}$ and X.

COROLLARY 2.4. *A cushioned pair qs-developable T_0-space is semi-developable, and hence is semi-metrizable.*

Proof. By the above theorem, a cushioned pair qs-developable T_0-space is stratifiable. A stratifiable space is perfect ([9], Theorem 2.2), and a perfect qs-developable space is semi-developable by the theorem 1.2. Recall that a semi-developable T_0-space is semi-metrizable ([1], Theorem 1.3).

3. Examples

EXAMPLE 3.1. There exists a qs-developable space which is neither quasi-developable nor semi-developable. Let X be the space of real line with the topology: Each irrational point is isolated and open intervals with rational end points are open [3]. Let Y be the *bow-tie region* space of Heath [12]. Let Z be the topological sum of X and Y. Clearly Z is qs-developable. Since semi-(or quasi-) developability is a hereditary property, Z is neither quasi-developable nor semi-developable.

EXAMPLE 3.2. There exists a first countable, perfect, paracompact, hereditarily Lindelöf and hereditarily separable space which is not elastic (and hence, not stratifiable, see [11] and is not qs-developable).

Consider the *Sorgenfrey line* (half open interval space, [13], 1K).

Assume there is a qs-development $\gamma=(\gamma_1, \gamma_2, \cdots)$ for the Sorgenfrey line X. Since $\{st\ (x, \gamma_n):\ n\in Z^+ \text{ and } x\in\gamma_n{}^*\}$ is a local base *at* x, for each x in X, there are positive integer $n(x)$ and a positive number $\varepsilon(x)$ such that

$$[x, x+\varepsilon(x))\subset st(x, \gamma_{n(x)})\subset[x, \infty).$$

Let $X_k=\{x\in X:\ n(x)=k\}$. Clearly $\bigcup X_k=X$. Since X is uncountable, there is an n such that X_n is uncountable. Let x and y be distinct points of X_n. We may assume $x<y$. If

$$[x, x+\varepsilon(x))\cap[y, y+\varepsilon(y))\neq\phi,$$

then $y\in[x, x+\varepsilon(x))\subset st(x, \gamma_n)$. This implies that $x\in st(y, \gamma_n)$.

But $st(y, \gamma_n)\subset[y, \infty)$ and $x\in st(y, \gamma_n)$ with $x<y$ are incompatible. This contradiction shows that $\{[x, x+\varepsilon(x)):\ x\in X_n\}$ is a disjoint collection of uncountably many intervals, which is impossible. This completes the proof that the Sorgenfrey line is not qs-developable.

EXAMPLE 3.3. There exists a non-developable (and hence, non-metrizable) cushioned pair qs-developable M_1-space. See [9] for the definition of M_1-space.

Let R' be the rational numbers. For $x\in R$, put $L_x=\{\langle x, y\rangle:\ \langle x, y\rangle\in R\times R,\ 0<y\}$ and $X=R\cup(\bigcup\{L_x{:}x\in R\})$. Then we will define a base for X as follows: For $s, t\in R'$ and $z=\langle x, w\rangle\in L_x$ such that $0<s<w<t$ we put $U^x_{s,t}(z)=\{\langle x, y\rangle:\ s<y<t\}$ and let $\mathcal{U}$ be the set of all such $U^x_{s,t}(z)$. For $r, s, t\in R'$ and $z\in R$ such that $s<z<t$ and $r>0$, we put

$$V_{r,s,t}(z)=(s, t)\cup(\{\langle w, y\rangle:\ 0<y<r,\ \ w\in(s, t)-\{z\}\}),$$

and let $\mathcal{V}$ be the set of all such $V_{r,s,t}(z)$. Now put $\mathcal{B}=\mathcal{U}\cup\mathcal{V}$.

Then $\mathcal{B}$ is a σ-closure preserving base making X into a nonmetrizable first countable M_1-space.

Now we will define a cushioned pair qs-development as follows:

For each $n\in Z^+$, $p\in R$ and $a\in R$, let

$$M_{npa}=\{\langle x, y\rangle\in X{:}y=a(x-p),\ \ (x-p)^2+y^2<1/n^2\},$$
$$N_{np}=\{\langle p, y\rangle\in X:\ 0<y<1/n\}.$$

Put

$$\gamma'_{s,t}=\{U^{z}_{s',t'}(z):x\in R\},\quad \textit{where } s'=\frac{2s+t}{3} \textit{ and } t'=\frac{s+2t}{3},$$

$$\delta'_{s,t}=\{U^{z}_{s,t}(z):x\in R\},$$

$$\gamma_n'=\{M_{(2n)pa}:\ p\in R,\ a\in R\}\cup\{N_{(2n)p}:\ p\in R\},$$

$$\delta_n'=\{M_{npa}:\ p\in R,\ a\in R\}\cup\{N_{np}:\ p\in R\}.$$

Now let φ be an 1-1 correspondence from Z^+ onto $R'\times R'$. Put

$$\gamma_{2n}=\gamma'_{\varphi(n)},\ \ \gamma_{2n-1}=\gamma'_n;\ \ \delta_{2n}=\delta'_{\varphi(n)},\ \ \delta_{2n-1}=\delta'_n.$$

We will define a cushioned pair qs-development (ξ,η):

Put $\xi_1=\gamma_1$. Assume ξ_{n-1} is defined. If $\xi_{n-1}^*\cap\gamma_n^*=\phi$, let $\xi_n=\xi_{n-1}\cup\gamma_n$. If $\xi_{n-1}^*\cap\gamma_n^*\neq\phi$. let $\xi_n=\gamma_n\cup\{C-\gamma_n^*:\ C\in\xi_{n-1}\}\cup\{C\cap cl(\gamma_n^*):\ C\in\xi_{n-1}\}$. Similarly we define $\eta_1=\delta_1$. Assume η_{n-1} is defined. If $\eta_{n-1}{}^*\cap\delta_n^*=\phi$, let $\eta_n=\eta_{n-1}\cup\delta_n$. If $\eta_{-1}^*\cap\delta_n^*\neq\phi$, let $\xi_n=\delta_n\cup\{D-\delta_n^*:\ D\in\eta_{n-1}\}\cup\{D\cap cl(\delta_n^*):\ D\in\eta_{n-1}\}$. One can verify that $\{\xi,\eta\}$ is a cushioned pair qs-development for X.

If X were developable, it would be metrizable since it is paracompact.

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