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Conformal Change in Einstein's $*g^{\lambda\nu}$ -Unified Field Theory. -II, The Vector S_λ

BY KYUNG TAE CHUNG

Summary. -In the first paper of this series, [2], we investigated how the conformal change enforces the connections and gave the complete relations between connections in Einstein's $*g^{\lambda\nu}$-unified field theory. In the current paper we wish to investigate how the vector

$$S_{\lambda\mu}{}^{\mu} \overset{\text{def}}{=\!=} S_\lambda$$

is transformed by the conformal change. This topics will be studied for all classes and all possible indices of inertia.

1. Auxiliary Results

This section contains some known results taken from [1] and [2] without proofs, which are needed in our subsequent considerations. The same abbreviations, notations, and terminologies will be used.

A) Consider two space-times X_4 and $\bar{X}_4$, on which the differential geometric structure is imposed by the general real tensors $*g^{\lambda\nu}$ and $*\bar{g}^{\lambda\nu}$ (2) respectively through the respective connections $\Gamma_{\lambda\mu}{}^{\nu}$ and $\bar{\Gamma}_{\lambda\mu}{}^{\nu}$ defined by

(1.1)a $$D_\omega {}^*g^{\lambda\nu} = -2\, S_{\omega\alpha}{}^{\nu}\ {}^*g^{\lambda\alpha},$$

(1.1)b $$\bar{D}_\omega {}^*\bar{g}^{\lambda\nu} = -2\, \bar{S}_{\omega\alpha}{}^{\nu}\ {}^*\bar{g}^{\lambda\alpha}.$$

We say that X_4 and $\bar{X}_4$ are *conformal* if, and only if

(1.2) $${}^*\bar{g}^{\lambda\nu}(x) = e^{-\Omega}\ {}^*g^{\lambda\nu}(x),$$

where $\Omega = \Omega(x)$ is an arbitrary function of position with at least two derivatives. This conformal change enforces a change of connection, and it can always be experessed as follows ([2], p. 204):

(1.3) $$\bar{\Gamma}_{\lambda\mu}{}^{\nu} = \Gamma_{\lambda\mu}{}^{\nu} + Q_{\lambda\mu}{}^{\nu} = \Gamma_{\lambda\mu}{}^{\nu} + M_{\lambda\mu}{}^{\nu} + N^{\nu}{}_{\lambda\mu},$$

where

$$M_{\lambda\mu}{}^{\nu} \overset{\text{def}}{=\!=} Q_{[\lambda\mu]}{}^{\nu}, \quad N_{\lambda\mu}{}^{\nu} \overset{\text{def}}{=\!=} Q_{(\lambda\mu)}{}^{\nu}.$$

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(2) Throughout the present paper, Greek indices take values I, II, III, IV unless explicitly stated otherwise and follow the summation convention, while Roman indices are used for the nonholonomic components of a tensor and run from 1 to 4. Roman indices also follow the summation convention with the exception of indices x, y, z, t.

B) (1.1)a can be reduced to ([2], p. 205-206)

(1.4)a $$M_{\alpha\beta\gamma}\ X^{\alpha\beta\gamma}_{\omega\mu\nu}=C_{\omega\mu\nu},$$

where

(1.4)b $$X^{\alpha\beta\gamma}_{\omega\mu\nu} \overset{\text{def}}{=\!=} \overset{000}{A}{}^{[\alpha\beta]\gamma}_{[\omega\mu]\nu}+\overset{110}{A}{}^{[\alpha\beta]\gamma}_{[\omega\nu]\nu}+2\overset{101}{A}{}^{[\alpha\beta]\gamma}_{[\omega\mu]\nu},$$

(1.4)c $$C_{\omega\mu\nu} \overset{\text{def}}{=\!=} \frac{1}{2}\ (H_{\omega\mu\nu}+3H_{\alpha[\mu\beta}{*k_{\omega]}}^{\nu}*k_{\nu}{}^{\beta})$$

(1.4)d $$H_{\omega\nu\mu} \overset{\text{def}}{=\!=} \Omega_{\nu}*k_{\mu\omega}+2\Omega_{\alpha}*k_{[\omega}{}^{\alpha}*h_{\mu]\nu}$$

(1.4)e $$\Omega_{\alpha} \overset{\text{def}}{=\!=} \partial_{\alpha}\Omega$$

For the last two classes, (1.4)a is equivalent to ([1], p. 210)

(1.5) $$2M_{\omega\mu\nu}=H_{\omega\mu\nu}+\Omega_{\alpha}{}^{(2)}*k_{\lambda}{}^{\alpha}\ *k_{\omega\mu} \text{ for the third class, and}$$
$$2M_{\omega\mu\nu}=\Omega_{\alpha}*k_{\mu\omega}+2\Omega_{\alpha}*k_{[\omega}{}^{\alpha}*k_{\mu]\lambda} \text{ for the fourth class.}$$

2. Conformal Change of the Vector S_{λ}

THEOREM (2.1)a. *(For the first two classes). The nonholonomic components of* $M_{\lambda} \overset{\text{def}}{=\!=} M_{\lambda\alpha}{}^{\alpha}$ *are*

(2.1)a $$M_{a}=M_{aba}+M_{aef}+M_{afe}$$

(2.1)b $$M_{e}=M_{efe}+M_{eab}+M_{eba}. \qquad (a, b=1, 2;\ e, f=3, 4)$$

Proof. Using the matrix equation (2.6) ([1], p. 1301), we have

$$\begin{aligned} M_{a}&=M_{ai}{}^{i}=M_{ab}{}^{b}+M_{ae}{}^{e}+M_{af}{}^{f}\\ &=M_{abi}\ *h^{ib}+M_{aei}\ *h^{ie}+M_{afi}\ *h^{if}\\ &=M_{aba}+M_{aef}+M_{afe}. \end{aligned}$$

The second relation may be obtained similarly.

THEOREM (2.1)b. *(For the first two classes). The nonholonomic components of* M_{λ} *are given by*

(2.2)a $$M_{i}=*\lambda_{i}\Omega_{i},$$

which is equivalent to

(2.2)b $$M_{\lambda}=\Omega_{\alpha}*k_{\lambda}{}^{\alpha}.$$

Proof. The non-holonomic components of $H_{\omega\mu\nu}$ are given by

$$H_{xyz}=\underset{y}{*\lambda}\Omega_{z}\ *h_{xy}+\underset{x}{*\lambda}\Omega_{x}\ *h_{yz}-\underset{y}{*\lambda}\Omega_{y}\ *h_{xz}\ {}^{(3)},$$

which is equivalent to

(2.3) $$H_{aef}=H_{efa}=\underset{a}{*\lambda}\Omega_{a},\ \ H_{fae}=H_{aef}=-\underset{a}{*\lambda}\Omega_{a}.$$

(3) This result may be obtained by substituting (2.6) ([1], p. 1301) into (1.4)d.

Since the type of two equations (1.4)a and (4.16) ([1], p. 1311) are similar, the non-holonomic components of $M_{\omega\mu\nu}$ are obtained from (5.2) ([1], p. 1313) as follows:

$$4 \underset{xyz}{*\lambda}\, M_{xyz} = (2 + \underset{z}{*\lambda}\, \underset{y}{*\lambda} + \underset{z}{*\lambda}\, \underset{x}{*\lambda}) H_{xyz} + \underset{z}{*\lambda}(\underset{x}{*\lambda} + \underset{z}{*\lambda})\, H_{zxy} + \underset{z}{*\lambda}(\underset{y}{*\lambda} + \underset{z}{*\lambda})\, H_{yzx}.$$

Substituting (2.3) into above, we have

$$4M_{aba} = \frac{2}{\underset{aba}{*\lambda}}\, H_{aba} = 0,$$

$$4M_{aef} = \frac{1}{\underset{aef}{*\lambda}}\, (2 + \underset{e}{*\lambda}\underset{f}{*\lambda} + \underset{a}{*\lambda}\underset{f}{*\lambda}) H_{aef} + \underset{f}{*\lambda}(\underset{a}{*\lambda} + \underset{f}{*\lambda}) H_{fae} = 2\underset{a}{*\lambda}\Omega_a, \tag{2.4}$$

$$4M_{afe} = \frac{1}{\underset{afe}{*\lambda}}\, (2 + \underset{e}{*\lambda}\, \underset{f}{*\lambda} + \underset{a}{*\lambda}\, \underset{e}{*\lambda})\, H_{fae} + \underset{e}{*\lambda}(\underset{a}{*\lambda} + \underset{e}{*\lambda})\, H_{aef} = 2\underset{a}{*\lambda}\Omega_a.$$

Hence, by (2.1)a and (2.4) we have $M_a = \underset{a}{*\lambda}\Omega_a$. Similarly we may easily see that (2.2)a holds for the case i=e.

THEOREM (2.2). *The vector S_λ is transformed by the conformal change* (1.2) *as*

$$\bar{S}_\lambda = S_\lambda + \Omega_\alpha * \lambda_\lambda{}^\alpha. \tag{2.5}$$

Proof. We have from (1.3)

$$\bar{S}_{\lambda\mu}{}^\nu = S_{\lambda\mu}{}^\nu + M_{\lambda\mu}{}^\nu$$

so that

$$\bar{S}_\lambda = S_\lambda + M_\lambda.$$

Theorem (2.1)b shows that (2.5) holds for the first two classes. On the other hand, since ${}^{(3)}*k_\lambda{}^\nu = 0$ for the last two classes, we have from (1.4)d and (1.5)

$$2M_\lambda = H_{\lambda\alpha}{}^\alpha + \Omega_\alpha{}^{(3)}*k_\lambda{}^\alpha = -\Omega_\alpha * k_\lambda{}^\alpha + 2\Omega_\alpha * k_{(\lambda}{}^\alpha * h_{\beta)}{}^\beta = 2\Omega_\alpha * k_\lambda{}^\alpha.$$

Hence, even in the last two classes we see that (2.5) holds.

References

[1] K. T. Chung, *Einstein's connection in terms of* $*g^{\lambda\nu}$, Nuovo Cimento, **27**, Serie X, 1963.
[2] K. T. Chung & M. S. Song, *Conformal change in Einstein's* $*g^{\lambda\nu}$*-unified field theory. -I*, Nuovo Cimento, Serie X, 58 B, 1968.

Yonsei University