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ON THE FOURIER-LAPLACE IMAGE OF SOME CLASSES OF FUNCTIONS AND DISTRIBUTIONS

BY SUNG KI, KIM

1. Introduction.

The Fourier-Laplace image of a class of C^∞-functions, any element of which can be estimated by $C\, exp(-\rho(x)\pm\varepsilon x)$ together with its derivatives, coincides with the class of entire functions $F(\xi+i\eta)$ which satisfies the estimate $|(\xi+i\eta)^N \partial_\zeta^m F(\xi+i\eta)| \leqslant C_{N,m} exp\rho^*(\eta)$ for all N and m, where $\rho^*(\eta)$ is the dual function of $\rho(x)$ in the sense of Young. On the other hand, the Fourier-Laplace image of $u\in\rho S$ is an entire function $F_u(\xi+i\eta)$ such that for any N, and m, there exists a constant $C_{N,m}$ with which we have

$$|\partial_\zeta^m F_u(\xi+i\eta)| \leqslant C_{N,m}(1+|\xi+i\eta|)^{-N} exp\rho^*(\eta)$$

and the converse holds [1]. Also the Fourier-Laplace image of $u\in{}_\rho S'$ is a holomorphic function $F_u(\xi+i\eta)$ in the tubular domain $R^n+i\Omega$ which satisfies the inequality

$$|F_u(\xi+i\eta)| \leqslant C_\eta(1+|\xi+i\eta|^{N_\eta}, \text{ for all } \eta \text{ in } \Omega$$

and the converse holds [3].
In this paper, we treat this problem in O_M and S', and obtain some results analogous to those in [1].

2. Preliminaries.

We shall use the following notations: $x=(x_1,\cdots,x_n)$, $\xi=(\xi_1,\cdots,\xi_n)$ and $\eta=(\eta_1,\cdots,\eta_n)$ are variable points in R^n; $\zeta=(\zeta_1,\cdots,\zeta_n)$ is a variable point in C^n and $\zeta=\xi+i\eta$; $\eta x=\eta_1x_1+\cdots+\eta_nx_n$; $\partial^k=\partial_1^{k_1}\cdots\partial_n^{k_n}$, $|k|=k_1+\cdots+k_n$, $k!=k_1!\cdots k_n!$ where $\partial_i=\dfrac{\partial}{\partial x_i}$ and $k=(k_1,\cdots,k_n)\in N^n$; $x^k=x_1^{k_1}\cdots x_n^{k_n}$, $\binom{k}{l}=\binom{k_1}{l_1}\cdots\binom{k_n}{l_n}$; $k\geqslant l$ means $k_i\geqslant l_i$ for all $i=1,\cdots,n$.

A function $\rho_i(x_i)$ defined on R is called strictly convex if, for any points ξ_i and η_i in R and all $t\in(0,1)$, the inequality

$$\rho_i(t\xi_i+(1-t)\eta_i) < t\rho_i(\xi_i)+(1-t)\rho_i(\eta_i)$$

is valid.

Let S be the space of rapidly decreasing C^∞-functions (i.e., tempered functions), O_M the space of slowly increasing functions, S' the space of tempered distributions and O_C' the space of rapidly decreasing distributions.

In this paper, $\rho(x)$ is in O_M and $\rho(x)=\sum_{i=1}^n\rho_i(x_i)$ where each $\rho_i(x_i)$ is strictly convex and $\rho_i(x_i)/|x_i| \to \infty$ as $|x_i| \to \infty$

The dual function $\rho_i^*(\eta_i)$ of $\rho_i(\eta_i)$ in the sense of Young is defined by $\rho_i^*(\eta_i)=$

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$\underset{x_i \in R}{\text{Max}}(-\rho_i(x_i)+\eta_i x_i)$. We denote by $\rho^*(\eta)$ the function $\sum_{i=1}^{n} \rho_i^*(\eta_i)$.

DEFINITION. Let E be the space S or O_M, E' the space S' or O_C' and Ω a subset of R^n.

$$\rho E=\{u(x)\in C^\infty(R^n)\,|e^{\rho(x)}u(x)\in E\}$$
$$_\Omega E=\{u(x)\in C^\infty(R^n)\,|e^{\lambda x}u(x)\in E \text{ for all } \lambda \text{ in } \Omega\}$$
$$\rho E'=\{u(x)\in D'(R_n)\,|e^{\rho(x)}u(x)\in E'\}$$
$$_\Omega E'=\{u(x)\in D'(R^n)\,|e^{\lambda x}u(x)\in E' \text{ for all } \lambda \text{ in } \Omega\}$$

For two subsets A and B of R^n such that $A\subset B$, $_AE\supset {}_BE$ and $_AE'\supset {}_BE'$. But we have the following

PROPOSITION. *If B is the convex hull of A, then $_AE={}_BE$ and $_AE'={}_BE'$.*

Proof. For any η in B, we have $\eta=\sum_{i=i}^{m} \lambda_i\eta_i$, $\sum\lambda_i=1$, $\lambda_i>0$, $\eta^i\in A$ [2]

The function $\alpha(x,\eta)=e^{\eta x}/\sum_{i=1}^{m} e^{\eta^i x}$ is bounded C^∞-function in x together with its derivatives. Let $u(x)$ be in $_AE$(or $_AE'$), then $e^{\eta x}u(x)=\alpha(x,\eta)\sum_{i=1}^{m} e^{\eta^i x}u(x)$ is in E (orE'). Therefore $u(x)\in {}_BE$ (or $_BE'$).

Thus we may assume Ω is a convex subset of R^n. Let $u(x)$ be in $_\Omega S$ (or $_\Omega S'$), then we can define the Fourier transform $F_u(\xi+i\eta)$ of $e^{\eta x}u(x)$ for all η in Ω. We shall call $F_u(\xi+i\eta)$ the Fourier-Laplace transform of $u(x)$ [1].

3. Estimates of the Fourier-Laplace transform of $_\rho O_M$.

By the definition, the C^∞-function $e^{\eta x-\rho(x)}$ is in $S\subset O_M$. Since any element of O_M is a multiplier in S [3], $_\rho O_M\subset {}_{R^n}S$. Therefore we can consider the Fourier-Laplace transform of an element of $_\rho O_M$.

THEOREM 1. *The Fourier-Laplace transform of u in $_\rho O_M$ is an entire function $F_u(\xi+i\eta)$ which satisfies the following condition; For any α and m in N^n, and ε in R^n, there exists a constant $C_{\alpha,m,\varepsilon}$ such that*

$$|\partial_\zeta^m F_u(\xi+i\eta)|\leqslant C_{\alpha,m,\varepsilon}(1+|\xi+i\eta|)^{-|\alpha|}e^{\rho^*(\eta+\varepsilon)}$$

where we take sgn ε_i=sgn η_i and $\varepsilon_i\neq 0$ for all $i=1,\cdots\cdots,n$.

Proof. For any k and m in N^n,

$$[-i(\xi+i\eta)]^k\partial_\zeta^m F_u(\xi+i\eta)=\int_{R^n}e^{\rho(x)}u(x)e^{-\rho(x)}(-ix)^m\partial_x^k e^{-i(\xi+i\eta)x}dx$$

By the the Leibnitz formula, each of the derivatives

$$|\partial_x^k[e^{\rho(x)}u(x)(-ix)^m e^{-\rho(x)}]|\leqslant \sum_{l\leqslant k}\binom{k}{l}|\partial_x^l[e^{\rho(x)}u(x)(-ix)^m]|\,|\partial_x^{k-l}e^{-\rho(x)}|$$
$$\leqslant \sum_{l\leqslant k}\binom{k}{l}C_{l,m}(1+|x|^2)^{|P_{l,m}|}e^{-\rho(x)} \text{ for some } P_{l,m} \text{ in } N^n$$
$$\leqslant C_{k,m}(1+|x|^2)^{|P_{k,m}|}e^{-\rho(x)} \text{ for some } P_{k,m} \text{ in } N^n.$$

Since $e^{\eta x-\rho(x)}$ belongs to S, integrating by parts

$$(-i)^{|k|}(\xi+i\eta)^k\partial_\zeta^m F_u(\xi+i\eta)=(-1)^{|k|}\int_{R^n}\partial_x^k\,[e^{\rho(x)}u(x)e^{-\rho(x)}\,(-ix)^m]e^{-i(\xi+i\eta)x}\,dx$$

Therefore we have

$$|(\xi+i\eta)^k\partial_\zeta^m F_u(\xi+i\eta)|\leqslant\int_{R^n}C_{k,m}(1+|x|^2)^{|P_{k,m}|}e^{-\rho(x)+\eta x}dx \quad (1)$$
$$\leqslant C_{k,m,\varepsilon}e^{\rho^*(\eta+\varepsilon)}$$

On the other hand,

$$(1+|\xi+i\eta|)^{|\alpha|}=\sum_{|P|\leqslant|\alpha|}\binom{|\alpha|}{|P|}|\xi+i\eta|^{|P|}\leqslant\sum_{|P|\leqslant|\alpha|}\binom{|\alpha|}{|P|}\sum_{|k|=|P|}\frac{|P|!}{k!}|(\xi+i\eta)^{k}| \tag{2}$$

Hence we have the desired inequality.

REMARK. For u in ${}_{\rho}S$, we can take $\varepsilon=0$ in the above inequality [1].

THEOREM 2. *If* $\lim_{|x_i|\to\infty}\rho_i(x_i)/|x_i|^{l}=\infty$ *for some* $l>1$, *then the Fourier-Laplace transform of* u *in* ${}_{\rho}O_M$ *is an entire function* $F_u(\xi+i\eta)$ *which satisfies the following condition;*
For any α *and* m *in* N^n, *there exist a constant* $C_{\alpha,m,l}$ *in* R *and* $p_{\alpha,m,l}$ *in* N^n *such that*

$$(1+|\xi+i\eta|)^{|\alpha|}|\partial_{\zeta}^{m}F_u(\xi+i\eta)|\leqslant C_{\alpha,m,l}(1+|\eta|^2)^{|P_{\alpha,m,l}|}e^{\rho^*(\eta)}.$$

Proof. Since there exist constants a_0, b_0 such that $\rho_i(x_i)\geqslant a_0|x_i|^{l}-b_0$,

$$\begin{aligned}-\rho_i(x_i)+\eta_i x_i&=-\rho_i(x_i)+|\eta_i||x_i| \quad\text{for } \eta_i x_i>0\\ &\leqslant -a_0|x_i|^{l}+|\eta_i||x_i|+b_0\\ &\leqslant -|x_i|+b_0 \quad\text{for } |x_i|\geqslant\left(\frac{1+|\eta_i|}{a_0}\right)^{\frac{1}{l-1}}=L_{\eta i}\end{aligned}$$

Denoting by $q_{k,m}(x_i,\eta_i)$ the function $(1+x_i^2)^{|P_{k,m}|}e^{\eta_i x_i-\rho_i(x_i)}$,

$$\begin{aligned}q_{k,m}&\leqslant(1+x_i^2)^{|P_{k,m}|}e^{-\rho_i(x_i)} \quad\text{for } \eta_i x_i\leqslant 0\\ &\leqslant(1+x_i^2)^{|P_{k,m}|}e^{-|x_i|+b_0} \quad\text{for } \eta_i x_i>0 \text{ and } |x_i|>L_{\eta i}\\ &\leqslant(1+L_{\eta_i}^2)^{|P_{k,m}|}e^{\rho_i^*(\eta_i)} \quad\text{for } \eta_i x_i>0 \text{ and } |x_i|\leqslant L_{\eta i}.\end{aligned}$$

From the inequality (1), we have

$$|(\xi+i\eta)^{k}\partial_{\zeta}^{m}F_u(\xi+i\eta)|\leqslant C_{k,m,l}(1+|\eta|^2)^{|P_{k,m,l}|}e^{\rho^*(\eta)}.$$

Thus we have the desired inequality from this inequality and (2).
We obtain the following converse result of Theorem 2.

THEOREM 3. *If* $F(\xi+i\eta)$ *is an entire function such that for any* α *and* m *in* N^n, *there exist constants* $C_{\alpha,m}$ *and* $p_{\alpha,m}$ *in* N^n *for which*

$$(1+|\xi+i\eta|)^{|\alpha|}|\partial_{\zeta}^{m}F(\xi+i\eta)|\leqslant C_{\alpha,m}(1+|\eta|^2)^{|P_{\alpha,m}|}e^{\rho^*(\eta)},$$

then there exists a u *in* ${}_{\rho}O_M$ *whose Fourier-Laplace transform equals to* $F(\xi+i\eta)$.

Proof. We can define a function $u(x)$ by

$$u(x)=\int_{R^n}e^{i(\xi+i\eta)x}F(\xi+i\eta)d\xi$$

where η is arbitrary.
Now, we must show that this function u belongs to ${}_{\rho}O_M$.

$$\begin{aligned}\partial_x^{l}[e^{\rho(x)}u(x)]&=\int_{R^n}\partial_x^{l}[e^{i(\xi+i\eta)x}e^{\rho(x)}]F(\xi+i\eta)d\xi\\ &=\sum_{j\leqslant l}\binom{l}{j}\int_{R^n}\partial_x^{l-j}e^{i(\xi+i\eta)x}\partial_x^{j}e^{\rho(x)}F(\xi+i\eta)d\xi\\ &=\sum_{j\leqslant l}\binom{l}{j}\int_{R^n}P_j(x)(\xi+i\eta)^{l-j}e^{i(\xi+i\eta)x+\rho(x)}F(\xi+i\eta)d\xi.\end{aligned}$$

Since we can take an integer k and a constant C such that

$$|p_j(x)|\leqslant C(1+|x|^2)^{k} \text{ for all } j\leqslant l,$$

$$\partial_x^{l}[e^{\rho(x)}u(x)]=\sum_{j\leqslant l}\binom{l}{j}\frac{p_j(x)}{(1+|x^2|)^{k}}\int_{R^n}e^{\rho(x)}(1+|x|^2)^{k}e^{i\zeta x}\zeta^{l-j}F(\xi+i\eta)d\xi$$

$$=\sum_{j\leq j} C_j e^{\rho(x)}\int_{R^n} e^{i\zeta x}(1-\Delta_\zeta)^k[\zeta^{l-j}F(\xi+i\eta)]d\xi$$
$$=\sum_{m,j\leq l} C_j e^{\rho(x)}\int_{R^n} e^{i\zeta x}P_m(\zeta)\partial_\zeta^m F(\xi+i\eta)d\xi$$

($P_m(\zeta)$ are polynomials of degree less than l.)

$$\leqslant \sum_{m,j\leq l} C_j e^{\rho(x)+\eta x+\rho^*(\eta)}(1+|\eta|^2)^{|P_{\alpha,m}|} C_{\alpha m}\int_{R^n}\frac{|P_m(\zeta)|}{(1+|\xi+i\eta|)^{|\alpha|}}d\xi.$$

Taking $\eta_i=\rho_i'(x_i)$ and $|\alpha|=|l|+n+1$,

$$|\partial_x^l[e^{\rho(x)}u(x)]|\leqslant C_l(1+|\rho'(x)|^2)^{|P_{\alpha,l}|}$$

which shows that $u(x)$ belongs to ${}_\rho O_M$.

Next, consider the multiplication of $h(x)$ in ${}_\rho O_M$ and $u(x)$ in ${}_\rho S'$.

THEOREM 4. *The Fourier-Laplace transform of hu is an entire function* $F_{hu}(\xi+i\eta)$ *which satisfies the following condition;*

For any ε *in* R^n, *there exist an* α *in* N^n *and a constant* C_ε *such that*

$$|F_{hu}(\xi+i\eta)|\leqslant C_\varepsilon(1+|\xi+i\eta|)^{|\alpha|}e^{\rho^*(\eta+\varepsilon)},$$

Proof. $F_{hu}(\xi+i\eta)=\langle e^{2\rho(x)}h(x)u(x),\ e^{-2\rho(x)-i(\zeta+i\eta)x}\rangle$.
Since $e^{2\rho(x)}h(x)u(x)$ is in S', $e^{2\rho(x)}h(x)u(x)=\partial_x^p[(1+|x|^2)^k f(x)]$ for some p and a bounded function $f(x)$ [3].

$$F_{hu}(\xi+i\eta)=\langle(1+|x|^2)^k f(x),\ (-\partial_x)^p e^{-2\rho(x)-i\zeta x}\rangle$$
$$=\int_{R^n}(1+|x|^2)^k f(x)(-\partial_x)^p e^{-2\rho(x)-i\zeta x}dx.$$

By the Leibnitz formula,

$$(-\partial_x)^p e^{-2\rho(x)-i\zeta x}=\sum_{l\leq p}P_l(x)\zeta^{p-l}e^{-2\rho(x)-i\zeta x}.$$

Hence we have

$$|F_{hu}(\xi+i\eta)|\leqslant\sum_{l\leq p}\int_{R^n}C_l(1+|x|^2)^{k+n_l}|\zeta^{p-l}|\,e^{-2\rho(x)+\eta x}\,dx$$
$$\leqslant C_\varepsilon(1+|\xi+i\eta|)^{|\alpha|}e^{\rho^*(\eta+\varepsilon)}.$$

REMARK. Let Ω contain 0.

i) For $h\in{}_\Omega O_M$ and $u\in{}_\Omega S$ (or ${}_\Omega S'$), $F_{hu}(\xi+i\eta)$ is the convolution of an element of O_C' and one of S (or S').

ii) For $h\in{}_\Omega O_C'$ and $u\in{}_\Omega S$ (or ${}_\Omega S'$), the Fourier transform of $h*u$ is the multiplication of an element of O_M and one of S (or S').

References

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Seoul National University