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1. The algebraic structure of the space of the rest function.

Let Ω be an open subset of R^n. We denoted by $C_0^\infty(\Omega)$ the space of infinitely smooth functions with compact supports in Ω. The element of $C_0^\infty(\Omega)$ is called a *test function*. $C_0^\infty(\Omega)$ becomes an algebra under usual multiplication and becomes an LF space under the inductive limit topology.

We shall use the usual definitions and terminologies for algebra as are given in Jacobson [4] and we refer elementary properties of $C_0^\infty(\Omega)$ to Treves [9]. When $p\in\Omega$, we set $M_p=\{f\in C_0^\infty(\Omega);f(p)=0\}$. When $f\in C_0^\infty(\Omega)$, supp f denotes the support of f.

PROPOSITION 1. *The ideal M of $C_0^\infty(\Omega)$ is maximal modular iff for some $p\in\Omega$, $M=M_p$.*

Proof. We shall use the theory of Banach Algebra (cf. Rickart [6] p. 123). Let $C_\infty(\Omega)$ be the space of continuous functions on the locally compact space Ω vanishing at ∞. $C_0^\infty(\Omega)$ is a self-adjoint subalgebra of $C_\infty(\Omega)$ seperating Ω where $f\in C_0^\infty(\Omega)$ is quasi-regular if inf $\{|1-f(w)|;w\in\Omega\}>0$.

Therefore $C_0^\infty(\Omega)$ has the carrier space of all maximal modular ideals topologically isomorphic to Ω. Since this isomorphism is given by $M_p \longrightarrow p$, our proposition follows.

Simillar proof proves

PROPOSITION 2. *Let K be a compact subset of Ω, and let $C_0^\infty(K)=\{f\in C_0^\infty(\Omega);$ supp $f\subset K\}$. An ideal M of $C_0^\infty(K)$ is maximal modular iff $M=\{f\in C_0^\infty(K);\ f(p)=0\}$ for some $p\in Int\ K$, the interior of K.*

The next proposition follows from the existance of C^∞ partitions of unity subordinated to a locally finite open covering for Ω.

PROPOSITION 3. *An ideal M of $C_0^\infty(\Omega)$ is maximal iff $M=M_p$ for some $p\in\Omega$.*

From the above propositions it follows

PROPOSITION 4. *For any ideal M of $C_0^\infty(\Omega)$ the followings are equivalent;*

1. *M is maximal,*
2. *M is maximal modular and*
3. *M is primitive.*

Since $C_0^\infty(\Omega)$ is not a radical algebra, we get the following proposition.

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PROPOSITION 5. *$C_0^\infty(\Omega)$ is semi-simple.*

PROPOSITION 6. *$C_0^\infty(\Omega)$ is a topological algebra under the inductive limit topology. In this case every maximal ideal is closed.*

Proof. If a net $\{f_\alpha\}$ converges to f in $C_0^\infty(\Omega)$, then there exists a compact set K in Ω such that f belongs to $C_0^\infty(K)$ and $\{f_\alpha\}$ converges to f *in* $C_0^\infty(K)$. Let g be an element in $C_0^\infty(\Omega)$ and β be a scalar. Then $f_\alpha g$, $f_\alpha+g$, βf belongs to C (supp $g\cup K$) and the corresponding nets converges to fg, $f+g$ and βf, respectively. Therefore $C_0^\infty(\Omega)$ is a topological algebra. To see that any maximal ideal is closed, it is enough to show that, for any compact K of Ω, $M\cap C_0^\infty(K)$ is closed in $C_0^\infty(K)$. But it is clear when $M\cap C_0^\infty(K)=C_0^\infty(K)$. When $M\cap C_0^\infty(K)$ is a maximal ideal in $C^\infty{}_0(K)$, it follows easily from the fact that $M\cap C_0^\infty(K)=\{f\in C_0^\infty(K)\cap M; f(p)=0\}$ for some p in Int K.

2. Sectional representation of distributions

When Ω is an open subset of R^n, the space of all continuous linear functional on the LF space $C_0^\infty(\Omega)$ is called the space of all distributions, and its element is called a *distribution*, which we shall denoted by $\mathcal{D}'(\Omega)$. General properties of $\mathcal{D}'(\Omega)$ is contained in Treves [9] and Schwartz [7]. $\mathcal{D}'(\Omega)$ with strong dual topology is not a Frechet space. In this section we shall imbed $\mathcal{D}'(\Omega)$ in a fiber of Frechet spaces and try to investigate how much $\mathcal{D}'(\Omega)$ is close to the Frechet space. We shall heavily depend on the method of fibrating a locally convex space on Banach spaces contained in Treves [10].

DEFINITION 1. Let i and $N_j{}^i$ $(j=0,1,2,\cdots)$ be positive integers. We set $B^i(N_0{}^i, N_1{}^i, N_2{}^i,\cdots)=\{f\in C_0^\infty(\Omega);$ *supp* $f\subset K_i$ and $q^i{}_j(f)\leqq N_j{}^i (j=0,1,2,\cdots)\}$. Here $\{K_j\}$ is a sequence of compact subset of Ω such that $K_i\subset Int\ K_{i+1}$ and $\bigcup_i K_i=\Omega$ and $q_j{}^i$ is a semi-norm on $C_0^\infty(K_i)$ such that $q_j{}^i(f)=supp\{\sum_{|k|\leqq j}|D^kf(x)|;\ x\in K_i\}$.

The following proposition follows from the definition of bounded sets.

PROPOSITION 1. *The family of $B^i(N_0{}^i, N_1{}^i, N_2{}^i,\cdots)$ as i and $N_j{}^i$ vary over integers forms a base of bounded sets of $C_0^\infty(\Omega)$.*

DEFINITION 2. A family of bounded sets $\{B_\alpha\}_{\alpha\in\Omega}$ such that $B_1\subset B_2\subset B_3\subset\cdots\subset B_\alpha\subset\cdots$ is called a *simple chain* of bounded sets.

When $\{C_\beta\}_{\beta\in\Gamma}$ is another simple chain of bounded sets, we say $\{C_\beta\}$ *refines* $\{B_\alpha\}$ provided that for any $\alpha\in\Omega$ there exists $\beta\in\Gamma$ such that $B_\alpha\subset C_\beta$.

PROPOSITION 2. *For any simple chain of bounded sets of $C_0^\infty(\Omega)$ there exists a simple chain of countable bounded sets which refines the given simple chain.*

Proof. Since $\{B^i(N_0{}^i, N_1{}^i, N_2{}^i,\cdots)\}$ forms a base for bounded sets of $C_0^\infty(\Omega)$, we may assume that the elements of the given simple chain are of the forms $B^i(N_0{}^i,\cdots)$.

Suppose that the net $\{i\}$ of indices i has no supremum. Then we can choose an increasing subsequence $\{i_1, i_2,\cdots\}$ refining the net $\{i\}$. Then $\{B^{ij}(N_0{}^{ij}, N_1{}^{ij},\cdots)\}_{j=1,2,\cdots}$ refines the given simple chain.

If $\{i\}$ has a supremum, let j be the first index of N such that $\{N_j{}^i\}$ has no supremum. Then we can choose an increasing subsequence $\{N_j{}^{i_1}, N_j{}^{i_2}, \cdots\}$ which refines $\{N_j{}^i\}$. Then $\{B^{i_k}(N_0{}^{i_k}, N_1{}^{i_k}, \cdots)\}_{k=1,2,\cdots}$ refines the given simple chain.

If all the indices have supremums, our proposition in trivially true.

Strong dual topology of $\mathcal{D}'(\Omega)$ is given by taking the polar of bounded sets of $C_0^\infty(\Omega)$ as open sets. Of course this topology is determined by the continuous semi-norms corresponding to the convex open neighberhoods of O. When p and q are semi-norms on $\mathcal{D}'(\Omega)$, $p \leqq q$ means $p(u) \leqq q(u)$ for all u's$\in \mathcal{D}'(\Omega)$. For a semi-norm p we set ker $p=\{u \in \mathcal{D}'(\Omega);\ p(u)=0\}$. If $\{p_i\}_{i\in\Omega}$ i an increasing net, we call $\{p_i\}_{i\in\Omega}$ a *simple chain* of semi-norms. Anoter simple chain of semi-norms $\{q_j\}_{j\in\Gamma}$ is said to *refine* $\{p_i\}_{i\in\Omega}$ iff for any $i\in\Omega$ there exist $j\in\Gamma$ such that $p_i \leqq q_j$.

The following proposition is the dual form of the Proposition 2.

PROPOSITION 3. *Let $\{p_i\}_{i\in\Omega}$ be a simple chain of continuous semi-norms on $\mathcal{D}'(\Omega)$. Then exists a simple chain of countable countinuous semi-norms which refines the given simple chain.*

DEFINITION 3. A simple chain composed of countable continuous semi-norms $\{p_1, p_2, \cdots\}$ is called a *Frechet spectrum* which we shall denote by p; i. e. $p=(p_1, p_2, \cdots)$.

When p and q are Frechet spectrums, $p \leqq q$ means that $p_i \leqq q_i$ for all $i=1, 2, 3, \cdots$. Let P the set of Frechet spectrums. If for any p and q in P there exist r in P such that $p \leqq r$ and $q \leqq r$, we say that P is *irreducible*. If $\bigcup_{p\in P} \bigcup_{p_i\in p} \{p_i\}$ is the base for all the continuous semi-norms on $\mathcal{D}'(\Omega)$, we say that P is *complete*. $\mathcal{D}'(\Omega)$ has complete irreducible families P of Frechet spectrums; for example, the set of all Frechet spectrums is complete and irreducible. Such a family is called a *spectrum* of $\mathcal{D}'(\Omega)$.

When P is a spectrum, $\bigcap_{p\in P} \bigcap_{p_i\in p} \ker p_i=0$. We shall set ker $p=\bigcap_{p_i\in p}\ker p_i$. In the sequel of this section E shall be $\mathcal{D}'(\Omega)$.

When $p=(p_1, p_2, \cdots)$ is a Frechet spectrum, the pseudometrizable space $\{E; p_1, p_2, \cdots\}$ will be denoted by $E_{(p)}$; i. e., $E_{(p)}$ is the space E equipped with the topology determined by the countable semi-norms $p_1, p_2, \cdots$. Let us set $E_{(p)}/\ker p=E_p$. Then E_p becomes a metrizable space.

The Frechet space obtained by completion of E_p shall be denoted by $\hat{E}_p$. Via the cannonical imbedding $E \Longrightarrow E_{(p)} \Longrightarrow E_p \Longrightarrow \hat{E}_p$, we get a continuous linear mapping w_p: $E \Longrightarrow \hat{E}_p$.

Let q be another Frechet spectrum such that $q \leqq p$. Then ker $p \subset \ker q$ and hence the identity mapping on E can be tranfered as a continuous linear mapping $w^q{}_p$: $\hat{E}_p \Longrightarrow \hat{E}_q$. We also get that if $r \leqq q \leqq p$, $w^p{}_r = w^p{}_r \cdot w^p{}_r$.

When p runs over a spectrum S, the disjoint union F of E_p becomes a fiber set on F. A *section* s on P is mapping from P into F such that $s(p) \in E_p$.

DEFINITION 4. When s is a section on P, s is called a *regular section* iff for any p and q in P, $p \leqq q$ implies $s(q)=w^p{}_q(s(p))$.

We shall denote by $\Gamma(P)$ the set of regular sections on P. $\Gamma(P)$ is a linear space. If we define that a net $\{s_\alpha\}$ of regular sections converges to s iff $\{s_\alpha(p)\}$ converge to $s(p)$ in $\hat{E}_p$ for all p in P, then $\Gamma(P)$ is a Hausdorff locally covex linear topological space.

In fact this topology is determined by semi-norms $s \longrightarrow \sup_{p \in A} p_i\ (s(p))$ where A is finite subset of P and i is an integer.

One can see easily that $\Gamma(P)$ is complete from the fact that $w^p{}_q$ is continuous. When u is an element in E, $p \longrightarrow w_p(u)$ is a regular section, which we shall denote by $w_p u$. From the definition of topology on $\Gamma(P)$, it follows that $u \longrightarrow w_p u$ is continuous. Moreover ker $w_p = \bigcap \{\ker p;\ p \in P\}$ and hence w_p is one to one.

PROPOSITION 4. *When P is a irreducible complete spectrum, $\mathcal{D}'(\Omega)$ and $\Gamma(p)$ are topologically isomorphic under w_p.*

Proof. We know that w_p is an one to one continuous linear mapping. We notice first that w_p^{-1} is continuous on $w_p(E)$ when $w_p(E)$ is equipped with a topology induced by the topology on $\Gamma(P)$. If a net $\{w_p(u_\alpha)\}$ converges to $w_p(u)$, then since for any p in P and for any i $p_i(u_\alpha - u) \longrightarrow 0$, $\{u_\alpha\}$ converges to u. This show that w^{p-1} is continuous. Since $\mathcal{D}'(\Omega)$ and $\Gamma(P)$ are complete Hausdorff spaces, to finish our proof it is enough to show that $w_p(\mathcal{D}'(\Omega))$ is dense in $\Gamma\ (P)$.

Let A be afinite subset of P and i be any integer. Since P is irreducible, there exists p in P such that for all p in A $q \geqq p$. Let s be any element in $\Gamma(P)$. Then for any $\varepsilon > 0$ there exist u in $\mathcal{D}'(\Omega)$ such that $p_i(s(p) - w_p(u)) < \varepsilon$.
But since $p \geqq q$,

$$q_i(w^q_p\, s(p) - w^p{}_q w^p(u)) \leqq p_i(s(p) - w_p(u)) < \varepsilon$$

This shows that $w_p(\mathcal{D}'(\Omega))$ is dense in $\Gamma(P)$.

3. Existance theorem of differential polynomials in $\mathcal{D}'^k(\Omega)$

Let Ω be an open subset of R^n and $P(D)$ be a differential polynomial on R^n. We can set

$$P(D) = \sum_v a_{v_1, v_2, \ldots, v_n} \ldots D_1{}^{v_1} D_2{}^{v_2} D_n{}^{v_n}$$
$$= \sum_v a_{v_1, v_2, \cdots v_n} \frac{\partial^{v_1}}{\partial x_1{}^{v_1}} \cdots \frac{\partial_{v_n}}{\partial x_{nv_n}}$$

where $a_{v_1, v_2, \cdots, v_n}$ are complex. We shall assume that the degree of $P(D)$ is t. Let $C_0{}^k(\Omega)$ be the space of C^k-functions with compact support in Ω. Let $\mathcal{D}'^k(\Omega)$ be the continuous dual of $C^k_0(\Omega)$. We shall denote by $\mathcal{E}'(\Omega)$ the space of distributions with compact support.

DEFINITION 1. Let u be the distribution. The *C^k-singular support* of u, which we shall denote by C^k-sing supp u, is the smallest closed set F such that u is C^k-function on $\Omega - F$.

In this section we shall generalize Hoermander's result [2] (Cf, Thm 3. 6. 4. Hoermander [3]). We shall give a sufficient condition on Ω such that $P(D)\ \mathcal{D}'^k(\Omega) = \mathcal{D}'^{k+t}(\Omega)$. In particular, the definition of C^k-strongly P convexity gives a little more concrete meaning of strongly P convexity.

We shall prove the following technical lemma.

LEMMA. *Consider the following 6 compact subsets of*
$H^{-1} \subset H^0 \subset H^1$ *and* $K^{-1} \subset K^0 \subset K^1$

and continuous semi-norms p on $C_0^{i+t}(\Omega)$ and q one $C_0^i(\Omega)$.
We shall assume the followings.

(1) K^{-1} is contained in the interior of K^0.

(2) For any u in $\mathcal{E}'(\Omega)$

supp $P(-D)u \subset H^{-1}$ implies supp $u \subset K^{-1}$ and
C^i-sing supp $P(-D)u \subset H^{-1}$ implies C^{i+t+1}-sing supp $u \subset K^{-1}$.

(3) For any $f \in C_0^i(\Omega)$

$$q(f) \geqq \sup_{x \in \Omega} |f(x)|.$$

(4) For any $f \in C_0^{i+t}(\Omega)$

supp $f \subset K^0$ implies $p(f) \leqq q(P(-D)f)$.

Under this assumptions for any $\varepsilon > 0$ there exists a continuous semi-norm q' on $C_0^i(\Omega)$ such that

(a) for any $f \in C_0^i(\Omega)$ $q'(f) \geqq q(f)$

(b) for any $f \in C_0^i(\Omega)$

supp $f \subset H^{-1}$ implies $q'(f) = (1+\varepsilon)q(f)$ and

(c) for any $f \in C_0^{i+t}(\Omega)$

supp $f \subset K'$ implies $p(f) \leqq q'(P(-D)f)$.

Proof. We shall follow Hoermander's proof in the case of $i=\infty$ and shall modify it patiently. Let Φ be the completion of $\{f \in C_0^{i+t}(\Omega);\ \text{supp } f \subset K^1\}$ with respect to semi-norms $q(P(-D)f)$ and $\sup_{x \in K} |D^k P(-D)f(x)|$ where k runs over multiindices such that $|k| = k_1 + k_2 + \cdots + k_n \leqq i$ and K over compact subsets of $U = \Omega - H^{-1}$. Clearly Φ is a Frechet space. Since $C_0^\infty(\Omega)$ is dense in $C_0^{i+t}(\Omega)$, there exists a constant C such that for any f in $C_0^{i+t}(\Omega)$ with supp $f \subset K^1$

$$\|f\|_{L^2} \leqq C \sup_{x \in \Omega} \|P(-D)f(x)\|.$$

Therefore by (3) Φ can be canonically imbedded in $L^2(K^1)$.

The linear map $f \longrightarrow P(-D)f$ maps $\{f \in C_0^{i+t}(\Omega)\ ;\ \text{supp } f \subset K^1\}$ into $C^i(\Omega)$ and with the restriction map of functions on Ω to functions on U induces a continuous linear map from Φ into $C^i(U)$.

Then, for all $u \in \Phi$, $P(-D)u$ is a C^i-function on U; that is, C^i-sing supp $P(-D)u \subset H^1$. Therefore by (2) u is a C^{1+t+1}-function on $V = \Omega - K^{-1}$. By the closed graph theorem the restriction of the domain Ω of the elements of Φ to V is a continuous linear map from Φ into $C^{i+t+1}(\Omega)$.

Let r be any semi-norm on $C^i(U)$. If r is continuous, then $q' = (1+\varepsilon)q + r$ satisfies always the conditions (a) and (b).

Let us now suppose that there is no such r satisfying the condition (c). Then there exists a sequence of functions in $C_0^{i+t}(\Omega)$, say $\{f_j\}$, such that supp $f_j \subset K^1$ and such that

$$p(f_j) \geqq 1+\varepsilon,\ q(P(-D)f_j) < 1 \text{ and } P(-D)f_j \longrightarrow 0 \textit{ in } C_0^i(U).$$

Clearly $\{f_j\}$ is bounded in Φ and hence in $L^2(K^1)$. Hence there exists f in $L^2(K^1)$ such that a sussequence of $\{f_j\}$ converges weakly to f in $L^2(K^1)$. On the other hand $P(-D)f_j \longrightarrow 0$ in $C^i(U)$. Hence $P(-D)f=0$ on U. By the first condition of (2) supp $f\subset K^{-1}$; i.e., $f=0$ on V. Since $J:\Phi \longrightarrow C^{i+t+1}(V)$ is continuous, $\{Jf_j\}$ is bounded in $C^{i+t+1}(V)$. By the Ascoli's theorem $\{Jf_j\}$ has a compact closure in $C^{i+t}(V)$ and from the above reasoning it follows that $Jf_j \longrightarrow 0$ in $C^{i+t}(V)$.

Let χ be an element in $C_0^\infty(K^0)$ such that χ is identically 1 on the neighborhood of K^{-1}. It exists due to (1).

Then $f_j'=(1-\chi)f_j \longrightarrow 0$ in $C^{i+t}(\Omega)$. Hence $P(-D)\ f_j' \Longrightarrow 0$ in $C^i(\Omega)$.

By the method of choosing $\{f_j\}$ it follows that for sufficiently large j, if we set $f_j''=\chi f_j$,

$$P(f_j'') > 1+2\varepsilon/3 \text{ and } q(P(-D)f_j'') < 1+\varepsilon/3.$$

Since supp $f_j''\subset K^0$, it contradicts to (4).

DEFINITION 2. Ω is *C^k-strongly P convex* iff for any compact subset K of Ω there exists a compact subset H of Ω such that for any u in $\mathcal{E}'(\Omega)$

supp $P(-D)u\subset K$ implies supp $u\subset H$ and

C^k-sing supp. $P(-D)u\subset K$ implies C^{k+t+1}-sing supp $u\subset H$.

THEOREM *If Ω is C^i-strongly P convex with respect to a defferential polynomial $P(D)$ of degree t, then $P(D)\ \mathcal{D}'^i(\Omega)=\mathcal{D}'^{i+t}(\Omega)$.*

Proof. Onec the lemma is established, the proof of Hoermander in the case $i=\infty$ can be modified without much difficulty. We omit the details.

References

[1] L. Ehrenpreis: *Solutions of some problems of division III.* Amer. J. Math. **78**, 1956.

[2] L. Hoermander: *On the range of differential and convolution operators.* Inst. for Adv. Study, Princeton, April, 1961, Mimeographed.

[3] ______ : *Linear partial differential operators.* Springer-Verlag, 1964.

[4] N. Jacobson: *Structure of rings.* Colloq. Pub. AMS, 1956.

[5] B. Malgrange: *Sur la propagation de la régularité des solutions des équations à coefficients constants.* Bull. Soc. Math. Phys. Roumanie, 1959.

[6] C. Rickart: *Banach algebras.* Van Nostrand, New York, 1960.

[7] L. Schwartz: *Théorie des distributions.* Hermann Paris, 1966.

[8] F. Trèves: *Linear partial differential equations with constant coefficients.* Gordon and Breach 1966.

[9] ______ : *Topological vector spaces, distributions and kernels.* Academic Press, 1967.

[10] ______ : *Locally convex spaces and linear partial differential equations,* Springer-Verlag, 1967.

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