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ON INVERSE LIMIT SYSTEMS AND CANTOR SET

by

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We know that the Cantor set is totally disconnected, compact, perfect, and metric. In this paper, we will make use of the results in the following lemmas and the concept of an inverse limit system to prove that every two such totally disconnected compact metric spaces are homeomorphic and hence will have a topological invariant property of the Cantor set.

Lemma 1. Let U be an arbitrary open covering of a metric space M and let n be an arbitrary integer. Then there is a refinement V of U composed of open sets with the property that each diameter is less than 1/n. If M is finite, then V can be taken to be finite.

Lemma 2. Let C be a component of a compact Hausdorff space and U be an arbitrary open set containing C. Then there is an open set V such that it contains C, is contained U, and has an empty boundary. Furthermore, the open set V is also closed.

Lemma 3. A one-to-one continuous mapping of a compact space X onto a compat space Y is a homeomorphism.

Theorem 1. If M is totally disconnected, compact, and metric, then there is a countable sequence U_1, U_2,......of finite coverings where each U_n is a collection of disjoint sets of diameter (less than 1/n) that are both open and closed and U_{n+1} is a refinement of U_n for each n.

Proof: Since a metric space is Hausdorff, from Lemma 2 we can obtain the results that if C is a component (a single point in this case) of M and if G is any open set containing C, then there is a closed and open set which contains C and is contained in G. From Lemma 1, we obtain the following results. Let U_0 be a covering of M. Each point of M belongs to some open set which is an element of U_0, since U_0 is a covering of M. If G_x is an element of U_0 containing x for each x in M, then there is a closed and open set H_x of diameter(<1) which contains x and is contained in G_x. And then $\{H_x\}$ is a covering of M and since M is compact, $\{H_x\}$ has a finite subcovering $\{H_i | i=1, 2, \ldots\ldots, n\}$, where each H_i is not necessarily disjoint. Consider the sets $G_1=H_1$, $G_2=H_2-H_1$, ..., $G_j=H_i-(\bigcup_{i=1}^{j-1}H_i)$. Then each G_i is open and closed, and each G_i is disjoint, for given G_i and G_j, $i<j$, G_i is a subset of H_i, and G_j is a subset of $M-H_i$. We have diameter $G_i \leq$ diameter$V_i<1$. Let $U_1=\{G_i\}$. By the general inductive step, it is obvious that we have a countable sequence U_1, U_2,......

Theorem 2. Let M be a compact totally disconnected metric space. Then M is homeomorphic to the inverse limit space of an inverse limit sequence of finite, discrete spaces.

Proof: Let U_1, U_2,......be the same sequence of coverings ae defined in the above Theorem 1. For each positive integer n, let U_n^* be the discrete topological space whose points are the open sets of U_n, i.e., U_n^* is the collection of sets U_n that has a discrete topology. Define a continuous mapping $f_n:U_n^* \to U_{n-1}^*$ $(n>1)$ as follows. If $G_{n,i}$ is an element of U_n, then there is a unique element $G_{n-1,j}$ of U_{n-1} containing $G_{n,i}$ because the elements of U_{n-1} are disjoint and U_n is a refinement of U_{n-i}. Let $f_n(G_{n,i})=G_{n-1,j}$ then each f_n is continuous since U_n^* is discrete. By the above definition, $\{U_n^*, f_n\}$ is an inverse limit sequence of compact, Hausdorff spaces, and then the inverse limit space U_∞ of $\{U_n^*, f_n\}$ is not empty. Next, we may define a mapping $h:U_\infty \to M$ which is a homeomorphism. If $p=(G_1, n_1, G_2, n_2,......)$ is a point of U_∞, then each U_i, n_i is a closed subset of M and G_i, n_i contains G_{i+1}, n_{i+1} for each positive integer i. Since M is compact and $\{G_i,n_i \mid i \text{ is natural}\}$ is a family of sets satisfying the finite intersection property, the intersection $\bigcap_{j=1}^{\infty} G_j$, n_j is not empty. Since the diameter of G_j, n_j is less than $1/j$, $\bigcap_{j=1}^{\infty} G_j$, n_j is a singleton set. Define this point as q. Let $h(p)=q$. Then h is one-to-one. For, if p is a point of U_∞, then $h(p)$ is in each of the point sets in M that are coordinate of p. Hence if two points p and p′ of U_∞ differ in the n th coordinate, then $h(p) \neq h(p')$ because the elements of U_n are disjoint. The mapping h is onto because $h(p)=q$ is in the intersection of such sequence of closed subsets of M. Note that the collection of sets $\{G_{j,i}\}$ is a basis for the topology of M. Since, for each $G_{j,i}$ of U_j, $h^{-1}(G_{j,i})$ consists of all points of U_∞ having $G_{j,i}$ for their j th coordinate and the point $G_{j,i}$ of U_j^* is open in U_j^*, $h^{-1}(G_{j,i})$ is open in U_∞. Thus h is continuous. By lemma 3, h is a hemeomorphism since U_∞ is compact, Hausdorff.

Theorem 3. Let G be an open set of a totally disconnected and perfect topological space. For each positive integer n, then G is a union of n disjoint nonempty open sets.

Proof: It is obvious for $n=1$. Suppose that it is true for $n=k$, i, e., $G=G_1 \cup G_2 \cup ...... \cup G_k$ where G_i is open, nonempty, and disjoint. Each G_k is not connected since the given space is totally disconnected and a single point is not open. Hence each G_k is decomposed into two disjoint open sets $G_{k,1}$ and $G_{k,2}$ and then $G_{k,1}$, $G_{k,2}$ are open in the given space since G_k is open in the space. Therefore, $G_1, G_2, ..., G_{k,1}$ $G_{k,2}$ is a decomposition of G for $n=k+1$. By the mathematical induction, the given assertion is true for any n.

Theorem 4. Any compact totally disconnected perfect metric space is homeomorphic to the Cantor set.

Proof: We know that the Cantor set is compact totally disconnected perfect metric space. Thus we are necessarily only to show that any two compact totally disconnected perfect metric spaces are homeomorphic. Let S and T be compact totally disconnected perfect metric spaces, and let U_1, U_2,......and V_1, V_2,......be countable sequences of coverings of S and T, respectively, where $U_k=\{G_{k,1},......,G_{k,n_k}\}$, $V_k=\{H_{k,1},......,H_{k,m_k}\}$, as produced in the proof of Theorem 2. If $n_1=m_1$, then we define $U'_1=U_1$ and $V'_1=V_1$. If $n_1>m_1$, then we may decompose $V_{1,1}$ into n_1-m_1+1 disjoint open (and closed) sets by using Theorem 3 and define $U'_1=U_1$ and $V'_1=\{V_{1,2}......, V_1, m_1\} \cup \{W_1, W_2, ..., W_{n_1-m_1+1}\}$, where $\{W_i\}$ is the decomposition of $V_{1,1}$. Similar method may be applied in the case $n_1<m_1$.

Assume that U'_j and V'_j have been defined so as to have the same number of elements.

Since the elements of $U'_j = \{G'_{j,1}, \ldots\ldots, G'_{j,n_j}\}$ are mutually disjoint closed sets, there is a natural number $m(>j)$ such that no set of diameter $1/m$ intersects any two $G'_{j,i}$. For V'_j, there is also a similar natural number m'. Let $m = \max\{m, m'\}$, then U_m is a refinement of U'_j and V_m is a refinement of V'_j. Consider the elements of U_m in $G'_{j,i}$ and the elements of V_m in $H'_{j,i}$ for each i. If there are the same number of these elements for a given i, then we leave them unaltered. But, for example, if there are more elements of U_m in $G'_{j,i}$ than elements of V_m in $H'_{j,i}$, then we use Theorem 3 to decompose one of the elements of V_m so as to have the same number of elements. By repeating such process we can obtain coverings U'_{j+1} and V'_{j+1} for each $i \leq n_j$, where U'_{j+1} and V'_{j+1} are refinements of U'_j and V'_j, respectively. For each i, the sets $G'_{j,i}$ and $H'_{j,i}$ have the same number of elements in U'_{j+1} and V'_{j+1}, respectively. Thus, by repeating the above process, we can obtain two countable sequences $U'_1, U'_2, \ldots\ldots$ and $V'_1 V'_2 \ldots\ldots$ Let $\overset{*}{U}_1, \overset{*}{U}_2, \ldots$ and $\overset{*}{V}_1, \overset{*}{V}_2, \ldots\ldots$ be the associated sequences of discrete topological spaces as defined in the proof of Theorem 2. We may define a mapping Φ: $\{\overset{*}{U}_n\} \to \{\overset{*}{V}_n\}$ by induction also. For $n=1$, let φ_1: $\overset{*}{U}_1 \to \overset{*}{V}_1$ be an arbitrary one-to-one correspondence. Assume that φ_{n-1}: $\overset{*}{U}_{1-n} \to \overset{*}{V}_{n-1}$ be defined as an arbitrary one-to-one correspondence, let φ_n: $\overset{*}{U}_n \to \overset{*}{V}_n$ be defined by assigning to each $\overset{*}{G}_{n,j}$ in $\overset{*}{U}_n$ an element of $\overset{*}{V}_n$ belonging $\varphi_{n-1}(f_n(G'_{n,j}))$, where f_n is the projection of U'_n into U'_{n-1}. Let $\Phi = \{\varphi_n\}$, then it is easy to see that Φ is a mapping of $\{\overset{*}{U}_n, f_n\}$ into $\{\overset{*}{V}_n, g_n\}$ and each φ_n is a homeomorphism. If $\varphi_\infty: U_\infty \to V_\infty$ is a mapping induced by Φ, then it is a homeomorphism and S is homeomorphic to T since, by Theorem2, U_∞ and V_∞ are homeomorphic to S and T, respectively.

References

1. Hocking, J.C., and Young, G.S., Topology. Addison-Wesley, Reading, Mass., 1961.
2. Kelley, J.L., General Topology. Van-Nostrand, New York, 1955.
3. Wallace, A.H., Algebraic Topology. W.A. Benjamin, Inc., New York, 1970.