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Correlation of L-Set, Open Set and Closed Set

by

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1. Introduction

The simplest sets of real numbers are the intervals. We define the open interval (a, b) to be the set $\{x; a<x<b\}$. We always take $a<b$ but we consider also the infinite intervals $(a, \infty)=\{x; a<x\}$ and $(-\infty, b)=\{x; x<b\}$.

Sometimes we write $(-\infty, \infty)$ for the set of all real numbers. We define the closed interval $[a, b]$ to be the set $\{x; a\leq x\leq b\}$. For closed intervals we take a and b finite. The half-open interval $(a, b]$ is defined to be $\{x; a<x\leq b\}$, and $[a, b)=\{x; a\leq x<b\}$. A generalization of the notion of an open interval is given by that of an open set.

In this dissertation, let us generally denote that the open set is G, the closed set is F, the general set is E, e and their complement are G', F', E', e' respectively.

properties

i) G' is closed set.

ii) F' is open set.

iii) $G-F$ is open set.

iv) $F-G$ is closed set.

v) Union of infinite numbers of open set is open set.

vi) Intersection of infinite numbers of closed set is closed set.

vii) Intersection of finite numbers of open set is open set.

viii) Union of finite numbers of closed set is closed set.

Proof: Let p be a limit point of G', then it is sufficient that we illustrate $p\in G$. If $p\overline{\in} G'$, then $p\in G$.

But since G is closed set, the very small neighborhood of p is the points of G. Thus our assumption that p is a limit point of G is contradiction. Therefore p belong to G'. Thus G' is closed set.

ii) Let p be arbitrary a point of F'. We take u which is very small ε-neighborhood of **p.** Then the point of F does not exist in u. Even if we take the smallest u there exists the **point** of F in u, then p is a limit point of F. But F is closed set, therefore $p\in F$. This show that $p\in$

F' is contradiction. Hence the all point in u is F', p is interior point of F'. Then since p is an arbitary point of F', F' is only interior point. Thus F' is open set.

iii) If $p \in G-F$, then $p \overline{\in} G-F$, $p \overline{\in} F$. Therefare $p \overline{\in} G \cap F'$.

Now, ε-neighborhood $u(p,\varepsilon)$ has the only element of $G \cap F'$. Hence p is interior point of $G \cap F'$. Then since $G \cap F' = G-F$, $G-F$ is set of only interior point, i.e. $G-F$ is open set.

iv) If p is a limit point of $F-G$, there exists infinite points of $F-G$ in neighborhood of $u(p,\varepsilon)$.

Now, from $F-G=FG'$, in the intersection of F and G', there exists an infinite number of point in neighborhood $u(p,\varepsilon)$.

But, because F, G' are closed sets, in $p \in F$ and $p \in G'$, $p \in FG' = F-G$.

Thus $F-G$ is closed set.

v) If $0_1, 0_2, \cdots\cdots, 0_n, \cdots\cdots$ are open set and their union is $M = 0_1 + 0_2 + \cdots\cdots + 0_n + \cdots\cdots$, then M is open set. Because if p is a point of M, p is contained within $0_1, 0_2, \cdots\cdots, 0_n, \cdots\cdots$.

For example, if p is contained in O_k, since O_k is open set, the very small neighborhood u of p is contained in O_k.

Thus since p is inner point of M, even though we take any point of M, it is inner point of M.

Thus M is open set.

vi) Let each set $F_1, F_2, \cdots\cdots, F_n \cdots\cdots$ are closed set and their product are denoted by $F = \prod_{i=1}^{\infty} F_i$ $(= \bigcap_{i=1}^{\infty} F_i)$. From Morgan's law, $F' = (\bigcup_{i=1}^{\infty} F_i)' = \bigcup_{i=1}^{\infty} F_i'$. Now by the property ii) $F_i'(i=1,2,\cdots\cdots)$ is open set. Therefore by property v), $\bigcup_{i=1}^{\infty} F_i'$ is open set. i.e. F' is open set and from $(F')'=F$, F is closed set.

vii) Let each set $0_1, 0_2, \cdots\cdots, 0_n$ are open sets and $G = \prod_{i=1}^{n} 0_i = (\bigcap_{i=1}^{n} 0_i)$. If $p \in G$ then $p \in 0_i$ $(i=1, 2, \cdots\cdots, n)$.

Since 0_1, 0_2, $\cdots\cdots$, 0_n are open set, the very small neighborhood of p is contained in $O_i(i=1, 2, \cdots\cdots, n)$ and in G too. Therefore p is inner point of G, thus G is open set.

viii) With respect to closed set, like vi) proved as duality by using complement. If each set $F_1, F_2, \cdots\cdots, F_n$ are closed set and $F = \bigcup_{i=1}^{n} F_i$, from the morgan's law, then $F' = (\bigcup_{i=1}^{n} F_i)' = \bigcap_{i=1}^{n} F'_i$.

Now since $F_i'(i=1,2,\cdots\cdots,n)$ is open set, by property vii), $\bigcap_{i=1}^{n} F_i'$ is open set i.e. F' is open set. Thus, from $(F')'=F$, F is closed set.

2. The improvement of Riemann-style measure.

Thorem 1: Open set G is denoted ω which ω is union of sequence of closed interval without in common inner point.

Proof: If we show that this theorem came into being in R^2, we know that there generally come into existance in R^n. For convenience sake, we make use of lattice net.

When $G^{(n)}$ is generated to lattice net depending on bisector of between segment of lattice net $G^{(n-1)}$, lattice net $G^{(1)}$, $G^{(2)}, \cdots\cdots, G^{(n)}, \cdots\cdots$ are called system of standard lattice.

Now, among the lattice net of $G^{(n)}$, if all of the contained in G is ω_n, $\omega_1 \subset \omega_2 \subset \cdots\cdots \subset \omega_n \subset \cdots\cdots$. Therefore $\{\omega_n\}$ is monotone increasing sequence $\omega = \bigcup_{n=1}^{\infty} \omega_n = \lim_{n\to\infty} \omega_n$.

If $x \in G$, because x is inner point, $U(x, \varepsilon) \in G$. Therefore with respect to the very large n, the interval which contained x is contained in G. Containing ω_n, also contained ω. i.e. from $x \in G$, $x \in \omega$. Thus $G \subset \omega'$. Now, in $\omega_n \subset G$, $\omega \subset G$. Thus $\omega = G$. But let's

$$\omega = \omega_1 + (\omega_2 - \omega_1) + \cdots\cdots + (\omega_n - \omega_{n-1}) + \cdots\cdots$$
$$= W_1 + W_2 + \cdots\cdots + W_n + \cdots\cdots,$$

then since $\omega_n - \omega_{n-1}$ is the union of the interval belong to $G^{(n)}$, $G = \sum_{i=1}^{\infty} W_i$. i.e. G is simple set of sequence of the interval.

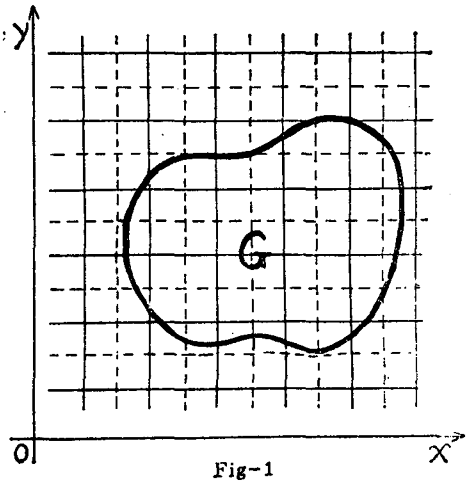

Fig-1

Note 1. From the proof of theorem, since open set G is the union of interval, G is L-set. By theorem 1, since the measure is $mG = \sum_{i=1}^{\infty} mW_i$, it is only inner product of Riemann-style measure.

Note 2. Since closed set F is compement of open set G, $G' \in L$. i.e. $F \in L$. Because $G \in L$, $G' = F$. Therefore closed set F is L-set, too.

Note 3. In lattice net $G^{(n)}$, the rest except that minor intervals contained in $F' = G$ has common points with F.

Let all of the points are $W^{(n)}$, $W^{(1)} \supset W^{(2)} \supset \cdots\cdots \supset W^{(n)} \supset \cdots\cdots$ (monotone decreasing sequence), and their intersection are F.

Indeed, $\lim_{n\to\infty} W^{(n)} = \bigcap_{i=1}^{\infty} W^{(i)} = F$, therefore $mF = \lim_{n\to\infty} W^{(n)}$, mF is only the outer product of Riemann-style measure.

Definition: Let the set of inner point of set s are open kernel. If the open kernel of the set s is denoted(s), (s) is open set. It is the largest open set contained in s.

Next, as the limit point of s, unless it belong to s, it is said that closure of set s which is union of it. If closure of set s is denoted [s], [s] is closed set and the smallest of closed set which contains s.

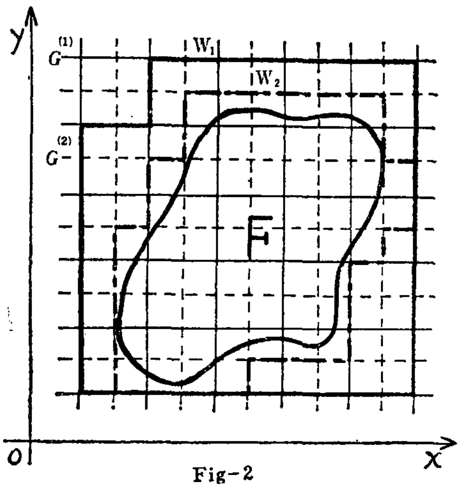

Fig-2

Note 4. With the respect to general set E, left open kernel of E is (E) and closure is [E], the intersection of all F such that $F \supset E \supset G$ is [E] and the union of all G are (E). Now by the Riemann-style measure, m[E] is the outer product of E and m(E) is the inner product of E.

As [E] − (E) is the boundary of set E, there exist m[E] − m(E). Because the approach of inner product and outer product are prevented and then the existance of Riemann-style measure is hard. To remedy, we suppose that a plan of improvement of Riemann-style measure.

By $G \supset E \supset F$, $\overline{m}E = \inf_{G \supset E} mG$ and $\underline{m}E = \sup_{F \subset E} mF$, the outer measure $\overline{m}E$ and the inner measure

$\underline{m}E$ of the general set E is defined, common value mE such that $\overline{m}E=\underline{m}E=mE$ is definition of measure E is a plan of improvement of the Riemann-style measure.

Theorem 2: The outer measure $\overline{m}E$ and the inner measure $\underline{m}E$ of Lebesque are as follows.

i) $\overline{m}E=\inf_{G\supset E} mG$

ii) $\underline{m}E=\text{sug}_{F\subset E} mF$

iii) If E is L-set, then $\inf_{G\supset E} mG=\sup_{F\subset E} mF$.

i.e. $mE=\overline{m}E=\underline{m}E$.

Proof: i) Since G is open set, in $G=\sum_{i=1}^{\infty}\omega_i$ (ω_i is interval), from definition of the outer measure

$$\overline{m}E=\inf_{E\subset \cup w_i} m\omega_i.$$

From $G=\sum\omega_i$, $mG=\sum m\omega_i$.

Now, since $\inf mG=mG\geq\inf_{E\subset\cup w_i}\sum m\omega_i$, $\overline{m}E\leq\inf mG$. Next, we prove that $\overline{m}E\geq\inf mG$.

We take $^{\forall}\varepsilon>0$, from definition there exists a sequence of the interval ω_i such that $E\subset \cup_i\omega_i$,

$\overline{m}E>\sum m\omega_i-\varepsilon$.

Now, let $\varepsilon=\sum_{i=1}^{\infty}\varepsilon_i$, $\varepsilon_i>0$ and we take the open interval G_i such that $G_i\supset w_i$,

$mG_i\leq m\omega_i+\varepsilon_i$. If $G=\cup_i G_i$, then G is the open interval, $G\supset E$ and $mG\leq\sum mG_i\leq\sum m\omega_i+\varepsilon$.

Therefore, $mG\leq\sum m\omega_i+\varepsilon=\overline{m}E+\varepsilon+\varepsilon$. i.e. $mG\leq\overline{m}E+2\varepsilon$. Now since ε is arbitrary from $\inf mG\leq\overline{m}E$, $\overline{m}E=\inf_{G\supset E} mG$.

ii) With the respect to the closed set belonged to E, let us prove it by principle of duality about complement.

From $E\supset F$, since $E'\subset F'$, $\overline{m}E'=\inf_{E'\subset F'} mF'$. Now the relation of the inner measure and the outer measure are $\overline{m}E'=m\omega-\underline{m}E$.

Hence $mF'=m\omega-mF$, (because F' is open set, $F'\in L$. That is, $\overline{m}F'=\underline{m}F'=mF'$ and since F is closed set, $F\in L$. Thus $\overline{m}F=\underline{m}F=mF$)

Therefore $\inf_{E'\subset F'} mF'=m\omega-\sup_{E\supset F} mF$,

$$\underline{m}F=\overline{m}E'-m\omega=\inf_{E'\subset F'} mF'-\inf_{E'\subset F'} mF'+\sup_{E\supset F} mF.$$

$$\text{i.e. } \underline{m}E=\sup_{E\supset F} mF$$

iii) By definition and i), ii) if $E\in L$ then $\overline{m}E=\underline{m}E=mE$ is clear.

3. Correlation of L-set, Open Set and Closed Set.

Theorem: The neressary and sufficient condition to be set E is L-set is arbitrary one of the following conditrions i), ii), iii).

i) When $^{\forall}\varepsilon>0$, there exists closed set F and open set G such that $G\supset E\supset F$, $m(G-F)<\varepsilon$.

ii) When $^{\forall}\varepsilon>0$, there exists general set e and F such that $E=F+e$, $\overline{m}e<\varepsilon$.

iii) When $^{\forall}\varepsilon>0$, there exists e and G such that $G=E+e$, $\overline{m}e<\varepsilon$

Proof: i) $\longrightarrow$; In theorem 2, $G\supset E\supset F$, $\overline{m}E=\inf_{E\supset E} mG$, and $\underline{m}E=\text{suf}_{F\subset E} mF$.

By the definition of inf, sup, since $mG \leq \overline{m}E$, $mF \geq \underline{m}E - \varepsilon\,(^{\forall}\varepsilon > 0)$,
$mG - mF \leq \overline{m}E - (\underline{m}E - \varepsilon) = \varepsilon$.

Now then, since $(G-F)+F=G$, $m(G-F)+mF=mG$.

i.e. $m(G-F)=mG-mF$.

Therefore $m(G-F)<\varepsilon$.

If $E \in L$, there exists G and F such that $G \supset E \supset F$, $m(G-F)<\varepsilon$.

$\longleftarrow$; If there exists G, F such that

$G \supset E \supset F$ and $m(G-F)<\varepsilon$, from

$mG-mF=m(G-F)$, $\inf\limits_{G\supset E} mG - \sup\limits_{E\supset F} mF \leq m(G-F) < \varepsilon$.

i.e. $\overline{m}E - \underline{m}E < \varepsilon$.

Also in $mE = m\omega - \overline{m}E' \leq m\omega$, $\underline{m}E \leq \inf\limits_{E\subset w} m\omega = \overline{m}E$.

Hence $\underline{m}E \leq \overline{m}E$. Thus $0 \leq \overline{m}E - \underline{m}E < \varepsilon$.

Now, When $^{\forall}\varepsilon>0$, $\overline{m}E=\underline{m}E=m\omega$. Thus $E \in L$.

After all $G \supset E \supset F$ and $m(G-F)<\varepsilon$ are the sufficient condition.

ii). $\longrightarrow$; Because there exrits G and F such that $G \supset E \supset F$, $m(G-F) > \varepsilon$ from i), let's put $E = F+e$, then $e=E-F \subset G-F$.

Thus $\overline{m}e \leq m(G-F) < \varepsilon$ (the reason that left side is $\overline{m}$ and right side is m is that e is arbitrary set, therefore the outer measure exists but are have no idea the existance of measure and since the right side $G-F$ is open set, $m=\overline{m}=\underline{m}$.)

Hence, if $E \in L$, $E=F+e$, $\overline{m}e<\varepsilon$ are the necessary condition.

$\longleftarrow$; If there exists F, e such that $E=F+e$, $\overline{m}e<\varepsilon$ from $E \supset F$,

$\underline{m}F=\overline{m}F=mF\leq\underline{m}E\leq\overline{m}E\leq mF+\overline{m}e$.

Thus $0\leq\overline{m}E-\underline{m}E\leq(\overline{m}F+me)-mF=\overline{m}e<\varepsilon$.

i.e. $0\leq\overline{m}E-\underline{m}E<\varepsilon$.

From $^{\forall}\varepsilon>0$, $\overline{m}E=\underline{m}E=mE$. i.e. $E\in L$. Therefor $E=F+e$, $\overline{m}e<\varepsilon$ are the sufficient condition.

iii) The necessary and sufficient condition to be $E \in L$ is that, there exists general set e and open set G such that $G=E+e$, $\overline{m}e<\varepsilon$.

This prove is to take complement and make use of ii). That is, $G'=(E+e)'=E'e'=E'-e$. Since $E'=G'+e$, by ii) there exists ε such that $\overline{m}e<\varepsilon$, and G' such that $E'=G'+e$. Now there is presented $E' \in L$ ($E'=G'+e$, $me<\varepsilon$) instead of condition $E \in L$, this is clear from the property of σ-system of L-Set. i.e. if $E \in L$, $E' \in L$ or if $E' \in L$, $(E')'=E \in L$.

Thus, that is clear.

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