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GENERALIZED INTEGRALS OF UNIVALENT FUNCTIONS

by

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1. Introduction

Let S denote the class of functions $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ that are analytic and univalent in the open unit disk $D=\{z:|z|<1\}$. Let K,S^*, and C denote respectively the subclasses of S whose members are convex, starlike relative to the origin, close-to-convex [1].

Let α be a point in n-dimensional real Euclidean space E^n. M.R. Ziegler [4] proved that for the functions $f_k(z)\in K(k=1,2,\ldots\ldots,n)$, the function defined by

$$g_\alpha(z)=\int_0^z\prod_{k=1}^n(f_k(t)/t)^{\alpha_k}dt,\quad \alpha=(\alpha_1,\ \cdots,\ \alpha_n)$$

is in K if α_k are positive and $\sum_{k=1}^n\alpha_k\leqq 2$. It is also a generalization of the result that Y.J. Kim and E.P.Merkes has proved in [2]. Moreover, the result is sharp. Sharpness here means that for each point α in E^n which is not restricted in the result, there exist function $f_k\in K$ such that the corresponding g_α is not in K.

In this paper, we investigate the univalence of functions of various integral types.

2. Convexity result.

We denote the functions g_α and G_α by the integral forms, respectively,

$$(1)\quad g_\alpha(z)=\int_0^z\prod_{k=1}^n(f_k(t)/t)^{\alpha_k}dt$$

and

$$(2)\quad G_\alpha(z)=\int_0^z\prod_{k=1}^n(f'_k(t))^{\alpha_k}dt$$

where $\alpha=(\alpha_1,\ \cdots,\ \alpha_n)\in E^n$, and each $f_k\in S$.

Suppose g_α and g_β (or G_α and G_β) are in one of the classes K and C. The first question that we consider is whether or not this implies g_α and g_β (or G_α and G_β) are in the same class for all the points in the line segment joining α and β in the E^n-space.

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Theorenl. The region in the E^n-space where the functions (2) are in one of the classes K or C for all choices of $f_k(k=1, 2, \ldots\ldots, n)$ in S is a convex region.

Proof. Suppose the function (2) is in K (or C) for two points α and β in E^n when f_k are given in S.

Define.

$$H_\lambda(z)=\int_0^z (G'_\alpha(t))(G'_\beta(t))^{1-\lambda}dt, \quad (0\leqq\lambda\leqq1).$$

By a theorem [2], we know that if G_α and G_β belong to K (or C), then the function H_λ is in K (or C).

However, we have

$$(3)\quad H_\lambda(z)=\int_0^z \prod_{k=1}^n (f_k'(t))^{\lambda\alpha_k+(1-\lambda)\beta_k}dt$$
$$=G_{\lambda\alpha+(1-\lambda)\beta},$$

where $\alpha=(\alpha_1, \cdots, \alpha_n)$ and $\beta=(\beta_1, \cdots, \beta_n)$ are in E^n.

The equality (3) means that if the functions G_α and G_β are in K(or C), then the function (2) belongs to the class K (or C) for all the points in the E^n-space joining these points by a line segment. This completes the proof of the theorem.

We remark here that the analog of Theorem 1 for functions of the form (1) can be proved by the similar methods to Theorem1.

3. The main theorems.

The problems to be considered in this section are the close-to-convexity of the functions defined by (1) and (2).

Note that

$$(4)\quad 1+\frac{zg_\alpha''(z)}{g'_\alpha(z)}=1-\sum_{k=1}^n\alpha_k+\sum_{k=1}^n\alpha_k\frac{zf_k'(z)}{f_k(z)}$$

and

$$(5)\quad Re\left\{1+\frac{zG_\alpha''(z)}{G'_\alpha(z)}\right\}=1-\sum_{k=1}^n\alpha_k+\sum_{k=1}^n\alpha_k\, Re\left\{1+\frac{zf_k''(z)}{f_k'(z)}\right\}$$

Theorem 2. Let f_k $(k=1, 2, \ldots, n)$ be in the class K. Then the function $g_\alpha\in K$ if $0\leqq\alpha_k\leqq2$ and $\sum_{k=1}^n\alpha_k\leqq2$.

Proof. First, take $\alpha_k\neq0$ and $\alpha_j=0$ $(j\neq k)$. Then

$$g_\alpha=\int_0^z (f_k(t)/t)^{\alpha_k}dt.$$

Hence for $\alpha_k\geqq0$

$$(6)\quad Re\left\{1+\frac{zg''_\alpha(z)}{g'_\alpha(z)}\right\}=1+\alpha_k+\alpha_k\, Re\left\{\frac{zf_k'(z)}{f_k(z)}\right\}$$

is nonegative if $0\leqq\alpha_k\leqq2$ since $Re\{zf_k'(z)/f_k(z)\}\geqq1/2$ for all $f_k\in K$.

Let us take the function $f_k = z/(1+z)$. Then $Re\{zf_k'(z)/f_k(z)\} = Re\{1/(1+z)\}$. Hence (6) does not hold when $\alpha > 2$ or $\alpha < 0$. This proves the boundaries $0 \leqq \alpha_k \leqq 2$ $(k=1, \ldots\ldots, n)$ are sharp.

Next, it follows from (4) that

$$Re\left\{1+\frac{zg''_\alpha(z)}{g'_\alpha(z)}\right.=1-\sum_{k=1}^{n}\alpha_k+\sum_{k=1}^{n}\alpha_k\ Re\left\{\frac{zf_k'(z)}{f_k(z)}\right\}$$

which is nonnegative provided $\sum_{k=1}^{n}\alpha_k \leqq 2$. For the sharpness, let $f_k = z/(1+z)$. Then

$$g_\alpha(z)=\int_0^z \frac{dt}{(1+t)\ \ \sum_{k=1}^{n}\alpha_k}$$

and by the second part of lemma 3 [2], this function is in K if and only if $0 \leqq \sum_{k=1}^{n}\alpha_k \leqq 2$. This completes the proof of theorem

Theorem 3. Let f_k $(k=1, 2, \ldots, n)$ be in K. Ten the function $g_\alpha \in C$ if α belongs to the convex region bounded by $-1 \leqq \alpha_1, \alpha_2, \cdots, \alpha_n \leqq 3$ for arbitrary choice of j and k $(j \neq k)$, $-1 \leqq \alpha_j + \alpha_k \leqq 3$, $\cdots$, and $-1 \leqq \sum_{k=1}^{n}\alpha_k \leqq 3$. The result is sharp.

Proof. We prove the theorem by induction. The theorem is true trivially when $n=1$ [3]. Assume that the theorem is satisfied in the $n-1$ dimensional case, i.e., without loss of generality we assume by symmetry that the theorem is true when α belongs to the convex region which is bounded by

$$\begin{aligned} & -1 \leqq \alpha_1, \cdots, \alpha_{n-1} \leqq 3 \\ (7)\quad & -1 \leqq \alpha_1+\alpha_2,\ \ \alpha_1+\alpha_3, \cdots, \alpha_{n-2}+\alpha_{n-1} \leqq 3 \\ & \vdots \\ & -1 \leqq \alpha_1+\cdots+\alpha_{n-1} \leqq 3 \end{aligned}$$

Combining together with the condition (7) and the condition in case which $n=1$, we have

$$\begin{aligned} & -1 \leqq \alpha_1, \cdots, \alpha_n \leqq 3, \\ & -1 \leqq \alpha_1+\alpha_2, \cdots, \alpha_{n-1}+\alpha_n \leqq 3, \\ & -1 \leqq \alpha_1+\alpha_2+\cdots+\alpha_n \leqq 3 \end{aligned}$$

For sharpness, for example, let us take a point $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_j, 0, \cdots, 0) \in E^n$ such that $-1 \leqq \alpha_1+\alpha_2+\cdots+\alpha_j \leqq 3$. Then the restrictions on $\alpha_1, \cdots, \alpha_j$ are established by setting $f_k(z) = z/(1+z)$ $(1 \leqq k \leqq j)$.

We also can obtain the following analogous theorems for functions of the form (1);

Theorem 4. Let f_k $(k=1, \cdots, n)$ be in S^*. Then

(i) the function $g_\alpha \in K$ if the point $\alpha \in E^n$ belongs to the convex region which is bounded by

$$0 \leqq \alpha_1, \cdots, \alpha_n \leqq 1,\quad \sum_{k=1}^{n}\alpha_k \leqq 1$$

(ii) the function $g_\alpha \in C$ if the point $\alpha \in E^n$ belongs to the convex region which is bounded by

$$\begin{aligned} & -1/2 \leqq \alpha_1, \cdots\cdots, \alpha_n \leqq 3/2 \\ & -1/2 \leqq \alpha_1+\alpha_2, \cdots, \alpha_{n-1}+\alpha_n \leqq 3/2 \\ & \vdots \\ & -1/2 \leqq \alpha_1+\alpha_2+\cdots+\alpha_n \leqq 3/2 \end{aligned}$$

In each case, the result is sharp.

Next, we consider the function G_α defined by (2).

Theorem 5. Let $f_k \in K$ $(k=1, \ldots, n)$. Then

(i) the function G_α in K when $\alpha \in E^n$ belongs to the closed convex region which is bounded by

(8) $0 \leqq \alpha_1 \cdots, \alpha_2, \ , \alpha \leqq 1, \ \sum_{k=1}^{n} \alpha_k \leqq 1$

(ii) the function G_α is in C when $\alpha \in E^n$ belongs to the closed convex region which is bounded by

$$
(9) \quad \begin{aligned}
&-1/2 \leqq \alpha_1, \cdots, \alpha_n \leqq 3/2, \\
&-1/2 \leqq \alpha_1+\alpha_2, \alpha_1+\alpha_3, \cdots, \alpha_{n-1}+\alpha_n \leqq 3/2 \\
&\vdots \\
&-1/2 \leqq \alpha_1+\alpha_2+\cdots+\alpha_n \leqq 3/2
\end{aligned}
$$

The results are sharp

Proof. (i) If $f_k \in K(k=1, \cdots, n)$, then $zf_k' \in S^*$. By Theorem 4 (i), we know that $G_\alpha \in K$ if and only if α belongs to the region is bounded by the condition (8). The sharpness follow by considering the function (2) obtained by the combining $z/(1+z)$ with itself or with z for the choices of f_k.

(ii) In the same way as (i), we have by Theorem 4 (ii), $G_\alpha \in C$ iff $\alpha \in E^n$ belongs to the region which is bounded by the condition (9). The sharpnesses come from considering the same functions as (i).

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