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A CHARACTERIZATION OF LOCAL COMPACTNESS

by

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It is shown that the cluster set of a net in a Hausdorff space X is a continuous function of the net if and only if X is locally compact. Our result should be compared with that of Fuller[3].

For a net (x_λ) in a topological space X the cluster set of (x_λ) is the set of all cluster points of (x_λ). we will denote the cluster set of (x_λ) by $\alpha(x_\lambda)$.

For a set X and subset U of X, $P(X)$ is the set of all nonempty subsets of X and $R(U, X)$ is the family of all nonempty subsets of X which intersect U.

If X is a topological space, the lower semifinite (lsf) topology on $P(X)$ has as a subbasis all sets of the form $R(U, X)$ where U is open in X.

We will denote all nets in a topological space X which have nonempty cluster sets by $X^\#$.

Let D be a directed set. The terminal set T_{λ_0} determined by $\lambda_0 \in D$ is $\{\lambda \in D \mid \lambda_0 \leq \lambda\}$.

The net space $X^\#$ will be assumed to carry the topology which has as a subbasis all sets of the form

$$\mathfrak{U}^\# = \{(x_\lambda) \in X^\# \mid {}^\forall T_\lambda,\ \varphi(T_\lambda) \cap U \neq \phi\},$$

where U is open in X and φ is net (x_λ) defined on D.

Theorem 1. The following four properties are equivalent

(1) X is locally compact.

(2) For each $x \in X$ and each neighborhood $U(x)$, there is a relatively compact open V with $x \in V \subset \overline{V} \subset U$.

(3) For each compact C and open $U \supset C$, there is a relatively compact open V with $C \subset V \subset \overline{V} \subset U$.

(4) X has a basis constisting of relatively compact open sets.

Proof: See the proof of XI 6.2 in [2].

Theorem 2. Let X be a Hausdorff space and $P(X)$ have (lsf) topology.

Then the cluster set function $\alpha: X^\# \to P(X)$ is continuous if and only if X is locally compact.

Proof: Assume X is locally compact. Let (x_λ) be in $X^\#$ and $R(V, X)$, where V is open in X, be a subbasic neighborhood of $\alpha(x_\lambda)$. Then for some p in $\alpha(x_\lambda)$, p is also in V. Hence there is a compact neighborhood U of p contained in V.

Consider the neighborhood of (x_λ), $\mathfrak{U}^\#$ and let (x_σ) be in this neighborhood. Then the net $(x_\sigma)_u$ consisting of the terms of (x_σ) in U is subnet of (x_σ). Since U is compact, $(x_\sigma)_u$ must have a cluster point x in U. So (x_σ) have a cluster point x in U.

Thus $\alpha(x_\sigma) \cap U \neq \phi$. Hence $\alpha(x_\sigma) \in R(V, X)$. Therefore α is continuous.

Assume X is not locally compact. There is then a point p in X such that no neighborhood of p is compact. Hence for every neighborhood U of p, there is a net (x_u) in U which has no cluster point.

Let (p) be net such that $x_\lambda = p$ for each $\lambda \in D$.

If $W^{\#}$, where W is open, is a subbasic neighborhood of (p). Then W is a neighborhood of p. Note $\alpha(p) = p$.

Now let $R(V, X)$, where V is open, be a subbasic neighborhood of $\alpha(p)$, so that V is a neighborhood of p.

For each neighborhood U of p, whenever $(x_u) = (x_{u\alpha}, x_{u\beta}, x_{u\gamma}, \cdots\cdots)$ and $x_0 \in X - V$, Consider the net $(x_u, x_0) = (x_0, x_{u\alpha}, x_{u\beta}, x_0, x_{u\gamma}, \cdots\cdots)$ which has cluster set $\alpha(x_u, x_0) = x_0$.

Let $\wedge$ be neighborhood system at p, Then the order relation $U_1 \leq U_2$ iff $U_2 \subset U_1$ directs $\wedge$. If $W^{\#}$ be a subbasic neighborhood of (p), $(x_u, x_0) \in W^{\#}$ for $U \subset W$. Thus the net of nets (x_u, x_0) converges to (p). But $\alpha(x_u, x_0)$ does not belong to $R(V, X)$ for any U. Therefore $\alpha: X^{\#} \rightarrow P(X)$ is not continuous.

References

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(2) James Dugundji, Topology, Allyn and Bacon, Inc, Boston, 1966.

(3) R.V.Fuller, A Characterization of Local Compactness, Proc. Amer. Math. Soc.37(1973), 615-616.