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ON FUNCTIONS STARLIKE WITH RESPECT TO SYMMETRIC POINTS

by

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1. Introduction

Let S be the class of functions $f(z)=z+a_2z^2+\cdots$ regular and univalent in the unit disc $E=\{z : |z|<1\}$ and let S^* be the subclass of functions starlike with respect to the origin. It is well known that $f(z)=z+a_2z^2+\cdots$ belong to S^* if and only if $Re\{zf'(z)/f(z)\}>0$ for $|z|<1$.

A few years ago M.S.Robertson[5] introduced a subclass of S consisting of functions $f(z)=z+a_2z^2+\cdots$ which satisfy the condition

(1.1) $Re\{zf'(z)[f(z)-f(-z)]^{-1}\}>0$ for $|z|<1$.

Such functions will be called here starlike with respect to symmctric points and corresponding subclass of S will be denoted by S^{**}.

M.S.Robertson has given in [5] a sufficient condition that a function $f(z)\in S^{**}$. This condition was stated in terms of subordination. In this paper we shall prove that Robertson's condition after a stight modification is also necessary. For the proof we need the following two lemmas.

2. An extension of Robertson's Theorem.

Lemma1. Suppose $\omega(z,t)=\sum_{n=1}^{\infty}b_n(t)z^n$ is regular in $|z|<1$ for $0\leqq t\leqq 1$.

Let $|\omega(z,t)|<1$ for $|z|<1$. $0\leqq t\leqq 1$, $\omega(z,0)\equiv z$. Let ρ be a positive real number for which

$$(2.1)\qquad \omega(z)=\lim_{t\to 0+}\left\{\frac{\omega(z,t)-z}{zt^{\rho}}\right\}$$

exists. Then

$$R_e\omega(z)\leqq 0 \text{ for } |z|<1.$$

If $\omega(z)$ is also regular in $|z|<1$ and $R_e\omega(0)\neq 0$, then

$$R_e\omega(z)<0 \text{ for } |z|<1.$$

Proof By Schwarz lemma we have for $|z|<1$. $|\omega(z,t)|\leqq|z|$ with equality only if $\omega(z,t)=z\exp\, i\theta(t)$, then the function

$$(2.2)\qquad \mu(z,t)=\frac{\omega(z,t)-z}{\omega(z,t)+z}$$

is regular and $Re\ \mu(z,t)<0$ for $|z|<1$. But when $\omega(z,t)=z\exp i\theta(t)$, $\mu(z,t)=i\tan(1/2\theta(t)$ is purely maginary. Thus $\mu(z,t)$ is regular and $Re\ \mu(z,t)\leqq 0$ in $|z|<1$ with equality occuring only if $\omega(z)=z\exp\ i\theta(t)$. For $t>0$ $|z|<1$ we may write

(2.3) $$Re\left\{\frac{\omega(z,t)-z}{zt^{\rho}}-\frac{2z}{\omega(z,t)+z}\right\}=Re\left\{\frac{2\mu(z,t)}{t^{\rho}}\right\}\leqq 0$$

(2.1) implies that $\lim_{t\to 0+}\omega(z,t)=z=\omega(z,0)$. Therefore, on letting $t\to 0$ in (2.3) we oftain $R_e\omega(z)\leqq 0$ for $|z|<1$. When $\omega(z)$ is also regular in $|z|<1$ and $R_e\omega(0)\neq 0$ we have further that $R_e\omega(z)<0$ in $|z|<1$. This follows since the maximum, in this case zero, of a non-constant harmonic function cannot occur at an interior point.

Using this Lemma 1. we shall prove

Lemma 2. Suppose $F(z,t)$ is regular in $|z|<1$ for $0\leqq t\leqq\delta$, $F(z,0)\equiv f(z)$, $f(z)\in S$ and $F(0,t)=0$ for $0\leqq t\leqq\delta$.

For each r, $0<r<1$, suppose that there exists $\delta(r)\in(0,\delta)$ such that for any t in $0<t\leqq\delta(r)$, $F(z,t)$ is subordinate to $f(z)$ in $|z|<r$ and the limit

$$\lim_{t\to 0+}\frac{F(z,t)-f(z)}{zt^{\rho}}=F(z)$$

exists for some $\rho>0$. Then

$$Re\left\{\frac{F(z)}{f'(z)}\right\}\leqq 0 \quad \text{for} |z|<1.$$

If in addition $F(z)$ is regular in $|z|<1$ and $ReF(0)\neq 0$ then

$$Re\left\{\frac{F(z)}{f'(z)}\right.<0 \quad \text{for } |z|<1$$

Proof. It follows from our assumption that there exists for any r, $0<r<1$, a function $\omega(z,t)$, regular in $|z|<r$ for each t, $0<t\leqq\delta(r)$ which satisfies the following conditions, $\omega(z,0)\equiv z$, $\omega(0,t)=0$ for all t, $0<t\leqq\delta(r)$ $|\omega(z,t)|\leqq r$ and $F(z,t)\equiv f\ \omega(z,t)\}$ for $|z|<r$ and $0<t\leqq\delta(r)$. Moreover, $\lim_{t\to 0+}\omega(z,t)=z=\omega(z,0)$.

Consider now

$$F(z)=\lim_{t\to 0+}\frac{F(z,t)-f(z)}{zt^{\rho}}=\lim_{t\to 0+}\frac{f\{\omega(z,t)\}-f\{\omega(z,0)\}}{zt^{\rho}}$$

We may assume that $\delta(r)$ is so small that for each t, $0<t\leqq\delta(r)$, we have $F(z,t)\not\equiv f(z)$. Otherwise $F(z)\equiv 0$ and there is nothing to prove. If $F(z,t)\not\equiv f(z)$ for any t, $0<t\geqq\delta(r)$, then $\omega(z,t)\not\equiv z$, hence by Schwarz's Lemma $|\omega(z,t)|<|\omega(z,0)|$ for $z\neq 0$ and we can write

$$F(z)=\lim_{t\to 0+}\frac{f\{\omega(z,t)\}-f\{\omega(z,0)\}}{\omega(z,t)-\omega(z,0)}=\lim_{t\to 0+}\frac{\omega(z,t)-\omega(z,0)}{zt^{\rho}}.$$

The first limit exists and so does the second limit. Thus Lemma 1 which is applied to the function $\omega(h,\tau)=r^{-1}\omega(rh,\ \delta(r))$, $|h|<1$, $0<\tau<1$ we see that

$$Re\ \omega(z)=Re\lim_{t\to 0+}\frac{\omega(z,t)-\omega(z,0)}{zt^{\rho}}\leqq 0 \quad \text{for } |z|<r$$

Hence $Re\dfrac{F(z)}{f'(z)}\leqq 0$ in $|z|<r$. Since r can be an arbitrary number of $(0,1)$, we have $Re\ \{F(z)/f'(z)\}\leqq 0$ in $|z|<1$. If $ReF(0)\neq 0$, then $Re\left\{\dfrac{F(0)}{f'(0)}\right\}=ReF(0)<0$.

If $F(z)$ is regular and $f'(z)\neq 0$ then $Re\{F(z)/f'(z)\}$ is harmonic and by the maximum principle $Re\{F(z)/f'(z)\}<0$ in $|z|<1$. Now we are able to prove

THEOREM A necessary and sufficient condition that $f(z)\in S^{**}$ when f is univalent and $f'(0)\neq 0$, is that for any r, $0<r<1$, there should exists $\delta(r)>0$ such that for each t, $0<t\leqq\delta(r)$ $(1-t)f(z)+tf(-z)$ is subordinate to $f(z)$ in $|z|<r$.

Proof. Sufficiency.

We apply Lemma 2 with $\rho=1$ and $F(z,t)=(1-t)f(z)+tf(-z)$.

Then $F(z)=\lim_{t\to 0+}(zt)^{-1}$

$[F(z,t)-f(z)]=-z^{-1}[f(z)-f(-z)]$

By Lemma2. we have

$$Re\{-[zf'(z)]^{-1}[f(z)-f(-z)]\}<0 \text{ for } |z|<1.$$

and this implies

$$Re\{zf'(z)[f(z)-f(-z)]^{-1}\}>0 \text{ for } |z|<1.$$

Necessity. Consider

$$v(z,t)=Re\left\{\frac{zF_z'(z,t)}{F_t'(z,t)}\right\}=Re\{-z[f'(z)-t(f'(z)+f'(-z))][f(z)-f(-z)]^{-1}.$$

Since $f\in S^{**}$, we have $v(z,0)<0$ in $|z|<1$. By the maximum principle for harmonic function we have

$$v(z,0)<-\varepsilon(r)<0 \quad \text{in } |z|<r.$$

By Continuity of $v(z,t)$ with respect to t we can find a positive $\delta(r)$ such that $v(z,t)<\frac{-1}{2}\varepsilon(r)<0$ for each t, $0\leqq t\leqq\delta(r)$, $|z|<r$. Now, by a result of Bielecki and Lewandowski [1], the inequality $Re\left\{\frac{zF_z'(z,t)}{F_t'(z,t)}\right\}<0$, $|z|<r$, means that the image of $|z|<r$ under $F(z,t)$ shrinks with increasing t. Therefore $F(z,t)$ is subordinate to $F(z,0)=f(z)$ in $|z|<r$ and this proves the necessity.

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