On Construction of Binary Number Associtaion Scheme Partially Balanced Block Designs

U. B. Paik*

1. Introduction

In a Balanced Factorial Experiments (BFE) with n factors F_1, $F_2, \cdots, F_n$ at m_1, m_2, $\cdots, m_n$ levels respectively, Shah [15] has considered the following association scheme: the two treatments are the $(p_1, p_2, \cdots, p_n)$th associates, where $p_i=1$ if the ith factor occurs at the same level in both treatments and $p_i=0$ otherwise; $\lambda_{(p_1, p_2, \ldots, p_n)}$ will denote the number of times these treatments occur together in a block. He has showed that a BFE is Partially Blanced Incomplete Block (PBIB) design with respect to the above asssociation scheme. Kurjian and Zelen [6] have proved that factorial designs possessing a Property A (a particular structure for their matrix NN') are factorially balanced.

Hinkelmann [4] has defined the EGD/$(2^n$-1)$-$PBIB without referring to Shah [15] nor to Kurjian and Zelen [6]. He has proved the uniqueness of the association scheme. Also he has obtained the eigenvalues of NN', its determinant, its Hasse-Minkowski invariants, c_p, and non-existence theorems. Note that Hinkelmann characterized the associate classes of the designs by the ordered plet $(\gamma_1, \gamma_2, \cdots, \gamma_n)$, where $\gamma_i=0$ if ith factor occurs at the same level in both treatments and $\gamma_i=1$ otherwise, *i.e.*, $(\gamma_1, \gamma_2, \cdots, \gamma_n)$ $=(1,1,\cdots,1)-(p_1, p_2, \cdots, p_n)$. Kshirsagar [5] has pointed out that BFE are EGD/$(2^n$-1)$-$PBIB defined by Hinkelmann [4] and possess Property A defined by Kurkjian and Zelen [6]. Also he has obtained the eigenvalues

---

* Professor of Statistics, Korea University. This work was supported by 1974 Research Grant of Ministry of Education, Republic of Korea Government.

and eigen-vectors of the C matrix($C=rI-1/k\ NN'$) of such designs in a much simpler form given by Hinkelmann.

Paik and Federer [10] have investigated on PBIB designs considered by Shah [15] without knowing the results of Hinkelmann [4] nor Kshirsagar [5]. In the paper [10], the block designs possessing Property A are designated as PA Type Block Designs and likewise, those two-way elimination of heterogeneity designs possessing both Properties A and B (introduced by Zelen and Federer [18]) are designated as PAB Type Rectangular Designs and obtained the eigenvalues and eigen-vectors of PA type and PAB type designs, and considered the efficiency of such designs.

Also, Paik and Federer [11] have designated the association scheme considered by Shah [15] as Binary Number Association Scheme (BNAS) and proved the following theorems.

Theorem 1.1. Every PA type block design is a BFE, and conversely.

Theorem 1.2. Every Balanced Factorial Incomplete Block Design is a PBIB with BNAS, and conversely. So every PA type Incomplete Block Design is a PBIB with BNAS, and conversely.

Theorem 1.3. Any n-ary Partially Balanced Block (NPBB) design having BNAS is a BFE and is a PA type block design, and conversely.

Theorem 1.4. If the design is an NPBB having BNAS with respect to rows and to columns, then the design is a Balanced Factorial Rectangular Experiment and is a PAB Type Rectangular Design.

Theorem 1.5. Every PAB Type Rectangular Design is an NPBB Rectangular Design having BNAS with respect to rows and to columns, and conversely.

Finally, Paik [9] has presented a practical method of intra- and inter-block analysis of PBIB having BNAS with various steps in computation and extended the method to the PAB Type Rectangular Designs.

In this paper, we investigate some construction methods associated with BNAS PBIB designs. We present and discuss the construction methods derived from the paper of Shah [16] in Section 2. Also, a particular method of construction of BNAS PBIB designs with two treatments per block is presented in Section 3.

2. Construction Methods of BNAS PBIB Designs Derived from the Paper of Shah [16]

Since a design uniquely determines its incidence matrix and vice versa, we may denote both a design and its incidence matrix by the same symbol. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be the designs expressed by their incidence matrices respectively, then

$$\mathcal{N}_{12}=\mathcal{N}_1\otimes\mathcal{N}_2$$

uniquely determines a design and so does $\mathcal{N}_{21}=\mathcal{N}_2\otimes\mathcal{N}_1$. Note that the designs $\mathcal{N}_{12}$ and $\mathcal{N}_{21}$ are structually the same, *i.e.*, one of them can be obtained from the other by simply renaming the treatments and renaming the blocks. The design $\mathcal{N}_{12}$ is called as Kronecker product of the designs $\mathcal{N}_1$ and $\mathcal{N}_2$ after Vartak [17]. The method is equivalent to the replacement of two elements, 0 and 1, by two matrices. A generalization of the idea is given by Shah [14], using only the incidence matrices of BIB designs for substitution. In the paper of Shah [16], same idea is extended to the case where substitution is by the incidence matrices of PBIB designs.

Definition 2.1. The s designs $\mathcal{N}_1$, $\mathcal{N}_2,\cdots,\mathcal{N}_s$, each in v treatments and b blocks, will be called associable designs, if there exists an orthogonal matrix L, such that $L'\mathcal{N}_i\mathcal{N}_j'L$ is diagonal for all $i,j=1,2,\cdots,s$. The matrix L will be called a canonical matrix of association.

Example 2.1. Any design $\mathcal{N}$ is associable with itself and its complementary design $J_{v\times b}-\mathcal{N}$, where $J_{v\times b}$ is a $v\times b$ matrix with all elements equal to unity.

Example 2.2. Any design is associable with a null design $O_{v\times b}$, or a randomized block design $J_{v\times b}$, provided the numbers of treatments and blocks are the same for the different designs, $O_{v\times b}$ is a $v\times b$ matrix with all elements equal to zero.

Example 2.3. The identity design is asssociable with any design where incidence matrix is a symmetric $v\times v$ matrix.

Definition 2.2. If there exists s $u\times w$ incidence matrices $\mathcal{N}_1^*, \mathcal{N}_2^*, \cdots, \mathcal{N}_s^*$, such that $\sum_{i=1}^{s} \mathcal{N}_i^* = J_{u\times w}$, and if there exists an orthogonal matrix L^* such that $L^{*\prime}(\mathcal{N}_i^*\mathcal{N}_j^{*\prime}+\mathcal{N}_j^*\mathcal{N}_i^{*\prime})L^*$ is diagonal for all $i,j=1,2,\cdots,s$, then the matix

$$A=\sum_{i=1}^{s} i\,\mathcal{N}_i^*$$

will be called a canonically balanced matrix in s integers $1,2,\cdots,s$.

Suppose there exists s BIB designs $\mathcal{N}_1^*, \mathcal{N}_2^*, \cdots, \mathcal{N}_s^*$, such that $\sum \mathcal{N}_i^* = J_{m\times n}$, and $(\mathcal{N}_i^*+\mathcal{N}_j^*)$ is also an incidence matrix of a BIB for all $i\neq j$, then $A=\sum i\ \mathcal{N}_i^*$ is a canonically balanced matrix in s integers $1,2,\cdots,s$.

Example 2.4. Let $\mathcal{N}_1^*=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$, $\mathcal{N}_2^*=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$, $\mathcal{N}_3^*=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$, then $\sum\mathcal{N}_i^*=$ $J_{3\times 3}$ and $\mathcal{N}_i^*\mathcal{N}_j^{*\prime}+\mathcal{N}_j^*\mathcal{N}_i^{*\prime}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ if $i=j$ and $\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$ if $i\neq j$ for $i,j=1,2,3$.

So, there exists an orthogonal matrix L^* such that $L^{*\prime}(\mathcal{N}_i^*\mathcal{N}_j^{*\prime}+\mathcal{N}_j^*\mathcal{N}_i^{*\prime})$ L^* is diagonal for all $i,j=1,2,3$. Therefore

$$A=\sum i\ \mathcal{N}_i^*=\begin{pmatrix}1&2&3\\2&3&1\\3&1&2\end{pmatrix}$$

is a canonically balanced matrix in 3 integers 1, 2, 3.

Given $v=m_1m_2\cdots m_n$, let $B_{(p_1,p_2,\cdots,p_n)}$ be an association matrix for the $(p_1, p_2,\cdots,p_n)$th associates, then it may be easily verified that $B_{(p_1,p_2,\cdots,p_n)}$ can be expressed as follows:

$$B_{(p_1,p_2,\cdots,p_n)}=\prod_{i=1}^{n}\otimes(J_{m_i}-I_{m_i})^{1-p_i},$$

where

$$(J_{m_i}-I_{m_i})^{1-p_i}=\begin{cases}J_{m_i}-I_{m_i}, & \text{if } p_i=0,\\ I_{m_i}, & \text{if } p_i=1.\end{cases}$$

Note $B_{(p_1,p_2,\cdots,p_n)}$ is a symmetric matrix, in which each row total and each column total is equal to the number of $(p_1,p_2,\cdots,p_n)$th associates, $n_{(p_1,p_2,\cdots,p_n)}$. Furthermore

$$\prod_{i=1}^{n}\otimes J_{m_i}=\prod_{i=1}^{n}\otimes[I_{m_i}+(J_{m_i}-I_{m_i})]$$

$$=\prod_{i=1}^{n}\otimes[(J_{m_i}-I_{m_i})^0+(J_{m_i}-I_{m_i})^1]$$

$$=\sum_{s=0}^{n}\left\{\sum_{p_1+p_2+\cdots+p_n=s}\prod_{i=1}^{n}\otimes(J_{m_i}-I_{m_i})^{1-p_i}\right\}$$

This implies

$$\sum_{s=0}^{n}\left\{\sum_{p_1+p_2+\cdots+p_n=s}B_{(p_1,p_2,\cdots,p_n)}\right\}=J_v.$$

Theorem 2.1. Let B_p be a association matrix for the $(p_1,p_2,\cdots,p_n)$th associates in BNAS, where $p=\sum_{h=1}^{n}p_h2^{n-h}$ for $v=m_1m_2\cdots m_n$, and let

$$N_p=B_p \quad \text{for} \quad p=0,1,\cdots,s;\ s=2^n\text{-}1,$$

then N_p for all $p=0,1,\cdots,s$, are 2^n mutually associable designs and $A=\sum_{p=0}^{s}(p+1)B_p$ is a canonically balanced matrix in 2^n integers $1,2,\cdots,2^n$.

The proof of the theorem is obvious from the fact that $\sum_p B_p=J_v$ and in BNAS,

$$B_pB_{p'}=B_{p'}B_p=\sum_{s=0}^{n}\left\{\sum_{p_1+p_2+\cdots+p_n=s}h(p_1,p_2,\cdots,p_n)\prod_{i=1}^{n}\otimes D_i^{p_i}\right\},$$

where $h(p_1,p_2,\cdots,p_n)$ are constants and

$$D_i^{p_i} = \begin{cases} I_{m_i}, & \text{if } p_i=0 \\ J_{m_i}, & \text{if } p_i=1. \end{cases}$$

Theorem 2.2. (Shah [16]) If there exists a canonically balanced matrix A in s integers $1, 2, \cdots, s$, given by $\sum_{j=1}^{s} i\ \mathcal{N}_i^*$ with the corresponding orthogonal matrix L^*, if there exists s mutually associable with incidence matrices $\mathcal{N}_1, \mathcal{N}_2, \cdots, \mathcal{N}_s$ with the canonical matrix of association equal to L, and if the integer c in A is replaced by the matrix $\mathcal{N}_c$ $(c=1, 2, \cdots, s)$, then the matrix A will be converted into an incidence matrix $\mathcal{N}$ of a design whose canonical matrix is $L^* \otimes L$.

Note that, from the method of construction, it follows that

$$\mathcal{N}=\sum_{i=1}^{s} \mathcal{N}_i^* \otimes \mathcal{N}_i. \tag{2.1}$$

If $\mathcal{N}_i\mathcal{N}_j'=\mathcal{N}_j\mathcal{N}_i'$, $\mathcal{N}\mathcal{N}'$ can be expressed as

$$\mathcal{N}\mathcal{N}'=\sum_{i=1}^{s} \mathcal{N}_i^*\mathcal{N}_i^{*\prime}\otimes\mathcal{N}_i\mathcal{N}_i'+\sum_{i<j}(\mathcal{N}_i^*\mathcal{N}_j^{*\prime}+\mathcal{N}_j'\mathcal{N}_i^{*\prime})\otimes\mathcal{N}_i\mathcal{N}_j'. \tag{2.2}$$

Therefore, if $\mathcal{N}_i^*\mathcal{N}_i^{*\prime}$, $\mathcal{N}_i\mathcal{N}_i'$ for all $i=1, 2, \cdots, s$, and $\mathcal{N}_i^*\mathcal{N}_j^{*\prime}, \mathcal{N}_j^*\mathcal{N}_i^{*\prime}, \mathcal{N}_i\mathcal{N}_j'$ for all $i<j$ have the Property A then $\mathcal{N}\mathcal{N}'$ also have the Property A. This means that the design $\mathcal{N}$ is a BNAS PBIB design.

In (2.1), if $\mathcal{N}_i^*\mathcal{N}_j^{*\prime}=\mathcal{N}_j^*\mathcal{N}_i^{*\prime}$, then $\mathcal{N}\mathcal{N}'$ may be expressed as

$$\mathcal{N}\mathcal{N}'=\sum_{i=1}^{s} \mathcal{N}_i^*\mathcal{N}_i^{*\prime}\otimes\mathcal{N}_i\mathcal{N}_i'+\sum_{i<j}\mathcal{N}_i^*\mathcal{N}_j^{*\prime}\otimes(\mathcal{N}_i\mathcal{N}_j'+\mathcal{N}_j\mathcal{N}_i'). \tag{2.3}$$

Example 2.5. Consider 3 associable designs

$$\mathcal{N}_1=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{N}_2=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{N}_3=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and the following canonically balanced matrix A in 3 integers 1, 2, 3;

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}, \quad i.e., \quad \mathcal{N}_1^*=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{N}_2^*=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{N}_3^*=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

In this case, $N_1^*N_1^{*\prime}=I_3$, $N_2^*N_2^{*\prime}=I_3$, $N_3^*N_3^{*\prime}=I_3$, $N_1N_1'=I_2$, $N_2N_2'=I_2$, $N_3N_3'=0_{2\times 2}$, $N_1^*N_2^{*\prime}+N_2^*N_1^{*\prime}=J_3-I_3$, and $N_1N_2'=N_2N_1'=J_2-I_2$, so $NN'=$
$2I_3\otimes I_2+(J_3-I_3)\otimes(J_2-I_2)$

$$=3I_6-I_3\otimes J_2-J_3\otimes I_2+J_6,$$

where

$$N=\begin{pmatrix} N_1 & N_2 & N_3 \\ N_2 & N_3 & N_1 \\ N_3 & N_1 & N_2 \end{pmatrix}=\begin{pmatrix} 1&0&0&1&0&0\\ 0&1&1&0&0&0\\ 0&1&0&0&1&0\\ 1&0&0&0&0&1\\ 0&0&1&0&0&1\\ 0&0&0&1&1&0 \end{pmatrix}.$$

Therefore, we obtain the following BNAS PBIB for $v=3\times 2$, $k=2$, $b=6$, $r=2$;

block	b_1	b_2	b_3	b_4	b_5	b_6
	1	2	2	1	3	4
	4	3	5	6	6	5

Example 2.6. Consider 2 associable designs

$$N_1=\begin{pmatrix} 1&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix},\quad N_2=\begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&1 \end{pmatrix},$$

and let $A=\begin{pmatrix} 1&2\\ 2&1 \end{pmatrix}$, *i.e.*, $N_1^*=\begin{pmatrix} 1&0\\ 0&1 \end{pmatrix}$, $N_2^*=\begin{pmatrix} 0&1\\ 1&0 \end{pmatrix}$.

In this case, we obtain the following BNAS PBIB for $v=2\times 3$, $k=2$, $b=6$, $r=2$.

$$N=\begin{pmatrix} 1&0&0&0&1&0\\ 0&0&1&1&0&0\\ 0&1&0&0&0&1\\ 0&1&0&1&0&0\\ 1&0&0&0&0&1\\ 0&0&1&0&1&0 \end{pmatrix},$$

$$NN'=3I_6-I_2\otimes J_3-J_2\otimes I_3+J_6.$$

Example 2.7. Let $N_1=\begin{pmatrix} 1&1&0&1\\ 1&1&1&0\\ 1&0&1&1 \end{pmatrix}$, $N_2=J_{3\times 4}-N_1$, and let $A=\begin{pmatrix} 1&2\\ 2&1 \end{pmatrix}$, *i.e.*,

$\mathcal{N}_1^*=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathcal{N}_2^*=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In this case, $\mathcal{N}_1^*\mathcal{N}_1^{*\prime}=I_2, \mathcal{N}_2^*\mathcal{N}_2^{*\prime}=I_2$, $\mathcal{N}_1^*\mathcal{N}_2^{*\prime}=\mathcal{N}_2^*\mathcal{N}_1^{*\prime}=J_2-I_2$, $\mathcal{N}_1\mathcal{N}_1'=I_3+2J_3$, $\mathcal{N}_2\mathcal{N}_2'=I_3$, and $\mathcal{N}_1\mathcal{N}_2'=\mathcal{N}_2\mathcal{N}_1'=J_3-I_3$. So,

$$\begin{aligned}\mathcal{N}\mathcal{N}' &= I_2\otimes(I_3+2J_3)+I_2\otimes I_3+(J_2-I_2)\otimes(J_3-I_3)\\ &=4I_6-2J_2\otimes I_3+2J_6.\end{aligned}$$

Therefore, we obtain the following BNAS PBIB for $v=2\times3$, $k=3$, $b=8$, $r=3$;

$$\mathcal{N}=\begin{pmatrix} 1&1&0&1&0&0&1&0\\ 1&1&1&0&0&0&0&1\\ 1&0&1&1&0&1&0&0\\ 0&0&1&0&1&1&0&1\\ 0&0&0&1&1&1&1&0\\ 0&1&0&0&1&0&1&1 \end{pmatrix}.$$

Example 2.8. Using the association matrices of BNAS for $v=2\times3$, let

$$\mathcal{N}_1=(J_2-I_2)\otimes(J_3-I_3),\ \ \mathcal{N}_2=(J_2-I_2)\otimes I_3,\ \ \mathcal{N}_3=I_2\otimes(J_3-I_3),$$

and let

$$A=\begin{pmatrix} 1&2&3\\ 2&3&1\\ 3&1&2 \end{pmatrix},\ \ i.e.,\ \ \mathcal{N}_1^*=\begin{pmatrix} 1&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix},\ \ \mathcal{N}_2^*=\begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&1 \end{pmatrix},\ \ \mathcal{N}_3^*=\begin{pmatrix} 0&0&1\\ 0&1&0\\ 1&0&0 \end{pmatrix},$$

then we obtain the follwing a BNAS PBIB design for $v=3\times2\times3$, $k=5$, $b=18$, $r=5$.

Note Since $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$ are the association matrices $\mathcal{N}_i\mathcal{N}_j'=\mathcal{N}_j\mathcal{N}_i'$ for all i,j and from Example 2.5, we know that $\mathcal{N}_i^*\mathcal{N}_j^{*\prime}+\mathcal{N}_j^*\mathcal{N}_i^{*\prime}=I_3$ or J_3-I_3 for all $i\neq j$.

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}	b_{17}	b_{18}
5	4	4	2	1	1	2	1	1	5	4	4	4	5	6	1	2	3
6	6	5	3	3	2	3	3	2	6	6	5	8	7	7	11	10	10
10	11	12	7	8	9	11	10	10	8	7	7	9	9	8	12	12	11
14	13	13	17	16	16	12	12	11	9	9	8	17	16	16	14	13	13
15	15	14	18	18	17	16	17	18	13	14	15	18	18	17	15	15	14

where the relationship between the treatment number t and the treatment combination (i_1, i_2, i_3) is $t=\left(\sum_{s=1}^{2}(\prod_{k=s+1}^{3} m_k)\ i_s\right)+i_3+1$. In this case, the treatment structure matrix $\mathcal{N}\mathcal{N}'$ is:

$$\mathcal{N}\mathcal{N}'=5I_{18}+I_3\otimes I_2\otimes J_3+J_3\otimes I_2\otimes J_3-2J_3\otimes I_2\otimes I_3-2I_3\otimes J_2\otimes J_3+2J_{18}.$$

Example 2.9. (Shah [16]) Confounding the interaction between two factors F_1 and F_2 in a 2^2-factorial, we get a BFE (=BNAS PBIB) with the incidence matrix $\mathcal{N}_1=\begin{pmatrix}1&0\\0&1\\0&1\\1&0\end{pmatrix}$. Let the balanced matrix A in 3 integers be given by

$$A=\begin{pmatrix}1&3&2\\2&1&3\\3&2&1\end{pmatrix}, \textit{i.e.}, \mathcal{N}_1^*=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \mathcal{N}_2^*=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, \mathcal{N}_3^*=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}.$$

Now, putting $\mathcal{N}_2=\mathcal{N}_3=J_{4\times 2}-\mathcal{N}_1$ and substituting for i in A, the matrix $\mathcal{N}_i$, $i=1, 2, 3$, we obtain a BFE in 3×2^2 in 6 blocks of 6 plots each.

Note that designs $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ are mutually associable and $\mathcal{N}_i^*\mathcal{N}_j^{*\prime}=\mathcal{N}_j^*\mathcal{N}_i^{*\prime}$ for all i, j.

Alternatively, if we take $\mathcal{N}_2=J_{4\times 2}-\mathcal{N}_1$ and $\mathcal{N}_3=0_{4\times 2}$, we obtain a BFE in 6 block of 4 plots each.

3. A Particular Method of Construction of BNAS PBIB Designs with Two Treatments per Block

In a BNAS PBIB design for $v=m_1m_2\cdots m_n$, suppose that $B_{(p_1, p_2, \cdots, p_n)}$ is the association matrix for $(p_1, p_2, \cdots, p_n)$th associates. If the treatment structure matrix $\mathcal{N}\mathcal{N}'$ has the following form

$$\text{(3.1)}\qquad \mathcal{N}\mathcal{N}'=n_{(p_1, p_2, \cdots, p_n)}\ I_v+B_{(p_1, p_2, \cdots, p_n)},$$

where $n_{(p_1, p_2 \cdots, p_n)}$ is a row total of $B_{(p_1, p_2, \cdots, p_n)}$, then we shall call this matrix as a basic treatment structure matrix associated with the $B_{(p_1, p_2, \cdots, p_n)}$.

Given an association matrix $B_{(p_1, p_2, \cdots, p_n)}=(b_{ij})_{(p_1, p_2, \cdots, p_n)}$, $i, j=1, 2, \cdots, v$, it may be easy to construct a basic BNAS PBIB design with two treatments per block, *i.e.*, write down the notations $b_{ij}(i<j)$, which have nonzero values such that, for example,

$$b_{12},\ b_{13},\ b_{17},\cdots,\ b_{23},\cdots\ b_{v-2,v},$$

then we can construct immediately the following basic BNAS PBIB design with two treatments per block,

block symbol	b_{12}	b_{13}	$b_{17}\cdots b_{23}\cdots$	$b_{v-2,v}$
	1	1	$1\cdots 2\cdots\cdots$	$v-2$
	2	3	$7\cdots 3\cdots\cdots$	v

and $r=n_{(p_1,p_2,\cdots,p_n)}$.

Thus, for any given association matrix $B_{(p_1,p_2,\cdots,p_n)}$, there always exists a basic BNAS PBIB design with two treatments per block. However, the design may be disconnected.

Let the notation $D(v,k,b,r)$ be a PBIB design such that number of treatments$=v$, block size$=k$, number of blocks$=b$, and number of replications $=r$, then we may obtain a design $D(v,k,b_1+b_2,\ r_1+r_2)$ by adding a design $D(v,k,b_2,r_2)$ to a design $D(v,k,b_1,r_1)$. In this case, if we denote $NN'(v,k,b_1+b_2,\ r_1+r_2)$, $NN'(v,k,b_2,r_2)$, and $NN'(v,k,b_1,r_1)$ as the corresponding treatment structure matrices to the above each design, respectively, then the following relationship holds:

$$(3.2)\qquad NN'(v,k,b_1+b_2,r_1+r_2)=NN'(v,k,b_1,r_1)+NN'(v,k,b_2,r_2).$$

Thus, a given v, $k=2$, r, a BNAS PBIB design $D(v,k=2,b,r)$ may be obtained as a linear combination of basic BNAS PBIB designs $BD(v,k=2,b_i,r_i)$, *i.e.*,

$$(3.3)\qquad D(v,k=2,b,r)=\sum_{i=1}^{s} z_i BD(v,k=2,b_i,r_i),\quad z_i\geq 0,$$

such that $r=\sum_{i=1}^{s} z_i r_i$, where $i=\sum_{h=1}^{n} p_h 2^{n-h}$, $s=2^n$. In this case, however, we must check whether this design is connected or not.

In an $m_1m_2\cdots m_n$-factorial, suppose $D(v,k,b,r)$ is a BNAS PBIB design, *i.e.*,

$$NN' = \sum_{s=0}^{n} \left\{ \sum_{\delta_1+\delta_2+\cdots+\delta_n=s} h(\delta_1, \delta_2, \cdots, \delta_n) \prod_{i=1}^{n} \otimes D_i^{\delta_i} \right\}, \tag{3.4}$$

then the efficiency factor associated with the estimate of generalized interaction $A_1^{x_1}A_2^{x_2}\cdots A_n^{x_n}$, where $x_i=0$ or 1 for $i=1, 2, \cdots, n$, is denoted by $\theta(x_1, x_2, \cdots, x_n)$ and is defined as follows (Kurkjian and Zelen [6]):

$$r\ \theta(x_1,x_2,\cdots,x_n) = \sum_{s=0}^{n-1} \left\{ \sum_{\delta_1+\delta_2+\cdots+\delta_n=s} g(\delta_1,\delta_2, \cdots, \delta_n) \prod_{i=1}^{n} E_i(x_i, \delta_i) \right\}, \tag{3.5}$$

for $g(0, 0, \cdots, 0) = r - \frac{1}{k}h(0, 0, \cdots, 0)$, $g(\delta_1, \delta_2, \cdots, \delta_n) = -\frac{1}{k}h(\delta_1,\delta_2,\cdots,\delta_n)$ if $(\delta_1, \delta_2, \cdots, \delta_n) \neq (0, 0, \cdots, 0)$ and $\neq (1, 1, \cdots, 1)$ and $E_i(x_i,\delta_i)$ are given by the table

		δ_i 0	1
x_i	0	1	m_i
	1	1	0

Let $\theta_1(x_1, x_2, \cdots, x_n)$ and $\theta_2(x_1, x_2, \cdots,x_n)$ be the efficiency factors for $v=m_1m_2\cdots m_n$ corresponding BNAS PBIB designs $D_1(v,k,b_1,r_1)$ and $D_2(v,k,b_2,r_2)$, respectively, and let $\theta^*(x_1,x_2, \cdots, x_n)$ be an efficiency factor corresponding BNAS PBIB design $D^*(v,k,b_1+b_2,r_1,+r_2)$, then, from (3.2), (3.4), and (3.5), we obtain the following relationship:

$$g^*(\delta_1,\delta_2,\cdots,\delta_n) = g_1(\delta_1, \delta_2, \cdots, \delta_n) + g_2(\delta_1, \delta_2, \cdots, \delta_n) \tag{3.6}$$

and

$$(r_1+r_2)\theta^*(x_1, x_2, \cdots, x_n) = r_1\theta_1(x_1, x_2, \cdots, x_n) + r_2\theta_2(x_1, x_2, \cdots, x_n). \tag{3.7}$$

Example 3.1. In the case of $v=2\times 3$,

$$B_{(0,0)}=(J_2-I_2)\otimes(J_3-I_3)=\begin{pmatrix}0&0&0&0&1&1\\0&0&0&1&0&1\\0&0&0&1&1&0\\0&1&1&0&0&0\\1&0&1&0&0&0\\1&1&0&0&0&0\end{pmatrix}$$

$$B_{(0,1)}=(J_2-I_2)\otimes I_3=\begin{pmatrix}0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\end{pmatrix}$$

$$B_{(1,0)}=I_2\otimes(J_3-I_3)=\begin{pmatrix}0&1&1&0&0&0\\1&0&1&0&0&0\\1&1&0&0&0&0\\0&0&0&0&1&1\\0&0&0&1&0&1\\0&0&0&1&1&0\end{pmatrix}$$

$$B_{(1,1)}=I_2\otimes I_3.$$

The nonzero values of $b_{ij}(i<j)$ in $B(0,0)$ are

$$b_{15},\ b_{16},\ b_{24},\ b_{26},\ b_{24},\ b_{35}.$$

So, the basic BNAS PBIB design $BD_1(6,2,6,2)$ associated with $B_{(0,0)}$ is as follows:

block symbol:	b_{15}	b_{16}	b_{24}	b_{26}	b_{34}	b_{35}
	1	1	2	2	3	3
	5	6	4	6	4	5

$$\mathcal{N}\mathcal{N}'(1)=2I_6+(J_2-I_2)\otimes(J_3-I_3)$$
$$=3I_6-J_2\otimes I_3-I_2\otimes J_3+J_6$$
$$2\theta_1(0,1)=3/2,\quad 2\theta_1(1,0)=2,\quad 2\theta_1(1,1)=1/2,$$

so this design is a connected one.

Likewise, $BD_2(6,2,3,1)$ associated with $B_{(0,1)}$ is

block symbol:	b_{14}	b_{25}	b_{36}
	1	2	3
	4	5	6

$$\mathcal{N}\mathcal{N}'(2) = I_6 + (J_2 - I_2) \otimes I_3$$
$$= J_2 \otimes I_3$$

$$\theta_2(0, 1) = 0, \quad \theta_2(1, 0) = 1, \quad \theta_2(1, 1) = 1,$$

so this is a disconnected design.

$BD_3(6, 2, 6, 2)$ associated with $B_{(1,0)}$ is

block symbol:	b_{12}	b_{13}	b_{23}	b_{45}	b_{46}	b_{56}
	1	1	2	4	4	5
	2	3	3	5	6	6

$$\mathcal{N}\mathcal{N}'(3) = 2I_6 + I_2 \otimes (J_3 - I_3)$$
$$= I_6 + I_2 \otimes J_3.$$

In this case, $2\theta_3(0, 1) = 3/2$, $2\theta_3(1, 0) = 0$, $2\theta_3(1,1) = 3/2$, so this is also a disconnected design.

If we wish a BNAS PBIB design $D(6, 2, 9, 3)$, we obtain such a design by adding $BD_2(6, 2, 3, 1)$ to $BD_1(6,2,6,2)$, *i.e.*,

block symbol:	b_{15}	b_{16}	b_{24}	b_{26}	b_{34}	b_{35}	b_{14}	b_{25}	b_{36}
	1	1	2	2	3	3	1	2	3
	5	6	4	6	4	5	4	5	6

In this case,

$$\mathcal{N}\mathcal{N}' = \mathcal{N}\mathcal{N}'(1) + \mathcal{N}\mathcal{N}'(2)$$
$$= 3I_6 - I_2 \otimes J_3 + J_6$$

$$3\theta(0, 1) = 2\theta_1(0, 1) + \theta_2(0, 1) = 3/2, \quad 3\theta(1, 0) = 2\theta_1(1, 0) + \theta_2(1, 0) = 3,$$
$$3\theta(1, 1) = 2\theta_1(1, 1) + \theta_2(1, 1) = 3/2.$$

Remark In the case of $v = 3 \times 2$,

$$B_{(0,0)} = (J_3 - I_3) \otimes (J_2 - I_2) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

So we obtain a basic BNAS PBIB design, $BD_4(6, 2, 6, 2)$ associated with

$B_{(0,0)}$ as follows:

block symbol:	b_{14}	b_{16}	b_{23}	b_{25}	b_{36}	b_{45}
	1	1	2	2	3	4
	4	6	3	5	6	5

$$NN'(4)=3I_6-I_3\otimes J_2-J_3\otimes I_2+J_6$$

$$2\theta_4(0,1)=2,\quad 2\theta_4(1,0)=3/2,\quad 2\theta_4(1,1)=1/2.$$

However, this design can be obtained by exchanging the names of the factors and renumbering the corresponding treatments in the design BD_1.

APPENDIX

We list all basic BNAS PBIB designs and their efficiency factors for $4 \leq v \leq 20$, $k=2$

v	m_1, m_2	(p_1,p_2) in B_p	b	r	$r\theta(0,1)$	$r\theta(1,0)$	$r\theta(1,1)$
4	2, 2	(0, 0)	2	1	1	1	0
		(0, 1)	2	1	0	1	1
		(1, 0)	2	1	1	0	1
6	2, 3	(0, 0)	6	2	3/2	2	1/2
		(0, 1)	3	1	0	1	1
		(1, 0)	6	2	3/2	0	3/2
9	3, 3	(0, 0)	18	4	3	3	3/2
		(0, 1)	9	2	0	3/2	3/2
		(1, 0)	9	2	3/2	0	3/2
10	2, 5	(0, 0)	20	4	5/2	4	3/2
		(0, 1)	5	1	0	1	1
		(1, 0)	20	4	5/2	0	5/2
14	2, 7	(0, 0)	42	6	7/2	6	5/2
		(0, 1)	7	1	0	1	1
		(1, 0)	42	6	7/2	0	7/2
15	3, 5	(0, 0)	60	8	5	6	7/2
		(0, 1)	15	2	0	3/2	3/2
		(1, 0)	30	4	5/2	0	5/2

v	m_1, m_2, m_3	(p_1, p_2, p_3) in B_p	b	r	$r\theta(x_1, x_2, x_3)$, (x_1, x_2, x_3) : (001)	(010)	(011)	(100)	(101)	(110)	(111)
8	2, 2, 2	(0, 0, 0)	4	1	1	1	0	1	0	0	1
		(0, 0, 1)	4	1	0	1	1	1	1	0	0
		(0, 1, 0)	4	1	1	0	1	1	0	1	0
		(0, 1, 1)	4	1	0	0	0	1	1	1	1
		(1, 0, 0)	4	1	1	1	0	0	1	1	0
		(1, 0, 1)	4	1	0	1	1	0	0	1	1
		(1, 1, 0)	4	1	1	0	1	0	1	0	1
12	2, 2, 3	(0, 0, 0)	12	2	3/2	2	1/2	2	1/2	0	3/2
		(0, 0, 1)	6	1	0	1	1	1	1	0	0
		(0, 1, 0)	12	2	3/2	2	3/2	2	1/2	2	1/2
		(0, 1, 1)	6	1	0	0	0	1	1	1	1
		(1, 0, 0)	12	2	3/2	2	1/2	0	3/2	2	1/2
		(1, 0, 1)	6	1	0	1	1	0	0	1	1
		(1, 1, 0)	12	2	3/2	0	3/2	0	3/2	0	3/2
18	2, 3, 3	(0, 0, 0)	36	4	3	3	3/2	4	1	1	5/2
		(0, 0, 1)	18	2	0	3/2	3/2	2	2	1/2	1/2
		(0, 1, 0)	18	2	3/2	0	3/2	2	1/2	2	1/2
		(0, 1, 1)	9	1	0	0	0	1	1	1	1
		(1, 0, 0)	36	4	3	3	3/2	0	3	3	3/2
		(1, 0, 1)	18	2	0	3/2	3/2	0	0	3/2	3/2
		(1, 1, 0)	18	2	3/2	0	3/2	0	3/2	0	3/2
20	2, 2, 5	(0, 0, 0)	40	4	5/2	4	3/2	4	3/2	0	5/2
		(0, 0, 1)	10	1	0	1	1	1	1	0	0
		(0, 1, 0)	40	4	5/2	4	5/2	4	3/2	4	3/2
		(0, 1, 1)	10	1	0	0	0	1	1	1	1
		(1, 0, 0)	40	4	5/2	4	3/2	0	5/2	4	3/2
		(1, 0, 1)	10	1	0	1	1	0	0	1	1
		(1, 1, 0)	40	4	5/2	0	5/2	0	5/2	0	5/2

REFERENCES

[1] Bose, R. C., Clatworthy, W. H., and Shrikhande, S. S., *Tables of Partially Balanced Designs with Two Associate Classes,* Technical Bulletin, No. 107, North Carolina Agricultural Experimental Station, 1954.

[2] Bose, R. C., and Mesner, D. M., "On Linear Associative Algebra Corresponding to Association Scheme of Partially Balanced Designs," *Annals of Mathematical Statistics, 30,* 21-28, 1959.

[3] Clatworthy, W. H., "Two Associate Classes and Two Treatments per Block," *The Journal of Research of the National Bureau of Standard, 54,* 177-190, 1955.

[4] Hinkelmann, K., "Extended Group Divisible Partially Balanced Incomplete Block Designs," *Annals of Mathematical Statistics, 35*, 681-695, 1964

[5] Kshirsagar, A. M., "Balanced Factorial Designs," *Journal of Royal Statistical Society, Series B, 28,* 559-567, 1966.

[6] Kurkjian, B. and Zelen, M., "Applications of the Calculus of Factorial Arrangements. I. Block and Direct Product Designs," *Biometrika, 50,* 63-73. 1963.

[7] Paik, U. B., "PAB type k-row by b-column Rectangular Designs (in Korean)," *Korea University Journal, Humanities and Social Science* 16, 215-248, 1970.

[8] Paik, U. B., "A Study on PA type Incomplete Block Designs and PAB type Rectangular Designs (in Korean)," *The Journal of Social Science,* Vol. 1, Korea University, 165-180, 1972.

[9] Paik, U. B., "Analysis of Partially Balanced Designs Having Binary Number Association Scheme," *The Journal of Social Science,* Vol. 2, Korea University, 85-131, 1973.

[10] Paik, U. B. and Federer, W. T., "On PA type Incomplete Block Designs and PAB type Rectangular Designs," *Journal of Royal Statistical Society, Series B, 35*, 245-251, 1972.

[11] Paik, U. B. and Federer, W. T., "Partially Balanced Designs and Properties A and B," *Communications in Statistics* Vol. 1, 331-350, 1973.

[12] Raghavavao, D., *Constructions and Combinatorial Problems in Design of Experiments,* John Wiley and Sons Inc., New York, 1971.

[13] Shah, B. V., "On Balancing in Factorial Experiments," *Annals of Mathematical Statistics*, 29, 766-799, 1958.

[14] Shah, B. V., "On Generalization of the Kronecker Product Designs," *Annals of Mathematical Statistics*, 30, 48-54, 1959.

[15] Shah, B. V., "Balanced Factorial Experiments," *Annals of Mathematical Statistics*, 31, 502-514, 1960.

[16] Shah, B. V., "A Matrix Substitution Method of Constructing Partially Balanced Designs," *Annals of Mathematical Statistics*, 31, 34-42, 1960.

[17] Vartak, M. N., "On an Application of Kronecker Product of Matrices to Statistical Designs," *Annals of Mathematical Statistics*, 26, 402-438, 1955,

[18] Zelen, M. and Federer, W. T., "Applications of the Calculus for Factorial Arrangements. II. Two Way Elimination of Heterogeneity," *Annals of Mathematical Statistics*, 35, 658-672, 1964.