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INTRINSIC PROBLEMS ON $S^n\left(\frac{1}{\sqrt{2}}\right)\times S^{n+1}\left(\frac{1}{\sqrt{2}}\right)$

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Dedicated to Prof. A. Kawaguchi on his seventieth birthday

§ 0. Introduction.

Ishihara and one of the present authors [2] have studied properties of an almost product structure in a Riemannian manifold, and have proved a theorem on the characterization of product spaces of two spheres:

THEOREM A. *Let (M,g) be a complete and connected Riemannian manifold of dimension m and let there be given in (M, g) two complementary almost product structures $P_i^{\ h}$ and $Q_i^{\ h}$ such that $\nabla_k P_i^{\ h}=0$. Assume that $P_i^{\ h}$ is of rank r and $2\leq r\leq m-2$. If there is in (M, g) a non-constant function λ satisfying*

$$P_j^{\ t} P_i^{\ s} \nabla_t \nabla_s \lambda = -\frac{\lambda}{a^2} P_{ji},$$

$$Q_j^{\ t} Q_i^{\ s} \nabla_t \nabla_s \lambda = -\frac{\lambda}{b^2} Q_{ji},$$

where a and b are positive constants, then (M, g) is isometric with $S^r(a)\times S^{m-r}(b)$ or $[S^r(a)\times S^{m-r}(b)]^*$, $[S^r(a)\times S^{m-r}(b)]^*$ being the factor space $S^r(a)\times S^{m-r}(b)/\sim$ with Riemannian metric induced from that of $S^r(a)\times S^{m-r}(b)$ by the projection.

Recently, the present authors and Suh [4] defined the so-called $(f, g, u_{(k)}, \alpha_{(k)})$-structure which is naturally induced on a hypersurface of a manifold with (f, g, u, v, λ)-structure or on a submanifold of codimension 2 of an almost contact metric space, and studied a hypersurface of even-dimensional sphere in terms of this structure by means of theorem A.

The main purpose of this paper is studying a characterization of $S^n\times S^{n+1}$ in terms of $(f, g, u_{(k)}, \alpha_{(k)})$-structure by using of Theorem A.

In § 1, we discuss intrinsic properties of $S^n\times S^{n+1}$. In § 2, we find some properties of $(f, g, u_{(k)}, \alpha_{(k)})$-structure induced on $S^n\times S^{n+1}$ as a submanifold of codimension 2 of $(2n+3)$-dimensional Euclidean space E^{2n+3} for later use. In § 3, we study complete Riemannian manifolds admitting an $(f, g, u_{(k)}, \alpha_{(k)})$-

structure which satisfies certain conditions.

§1. $S^n\left(\frac{1}{\sqrt{2}}\right)\times S^{n+1}\left(\frac{1}{\sqrt{2}}\right)$ as a submanifold of codimension 2 of E^{2n+3}.

Let E^{n+2} be an $(n+2)$-dimensional Euclidean space and 0 the origin of a cartesian coordinate system in E^{n+2} and denote by X the position vector of a point P in E^{n+2} with respect to the origin 0.

We consider a sphere $S^{n+1}\left(\frac{1}{\sqrt{2}}\right)$ with the center at 0 and with radius $\frac{1}{\sqrt{2}}$, and suppose that $S^{n+1}\left(\frac{1}{\sqrt{2}}\right)$ is covered by a system of coordinate neighborhoods $\{U ; x^a\}$, where here and in the sequel the indices $a, b, c\cdots$ run over the range $\{1, 2, \cdots, n+1\}$. Then the position vector X of a point P on $S^{n+1}\left(\frac{1}{\sqrt{2}}\right)$ is a function of (x^a) satisfying $X\cdot X=\frac{1}{2}$, where the dot denotes the inner product of two vectors in a Euclidean space.

Now we put

$$X_b=\partial_b X,\quad M=-\sqrt{2}X,\quad g_{cb}=X_c\cdot X_b, \tag{1.1}$$

where $\partial_b=\partial/\partial x^b$, and denote by ∇_c the operator of covariant differentiation with respect to the Christoffel symbols $\left\{{a \atop cb}\right\}$ formed with the metric tensor g_{cb}. Then X_b being tangent to $S^{n+1}\left(\frac{1}{\sqrt{2}}\right)$ and M being the unit normal to $S^{n+1}\left(\frac{1}{\sqrt{2}}\right)$, we have equations of Gauss and those of Weingarten respectively in the forms

$$\nabla_c X_b=\sqrt{2}g_{cb}M,\quad \nabla_c M=-\sqrt{2}X_c. \tag{2.2}$$

We next suppose that $S^n\left(\frac{1}{\sqrt{2}}\right)$ is covered by a system of coordinate neighborhoods $\{V ; (x^r)\}$, where here and in the sequel the indices $r, s, t, \cdots$ run over the range $\{n+2, \cdots, 2n+1\}$. Then the position vector Y of a point Q on $S^n\left(\frac{1}{\sqrt{2}}\right)$ is also a function of (x^r) satisfying $Y\cdot Y=\frac{1}{2}$.

We now put

$$Y_s=\partial_s Y,\quad N=-\sqrt{2}Y,\quad g_{ts}=Y_t\cdot Y_s, \tag{1.3}$$

where $\partial_s=\partial/\partial x^s$, and denote by ∇_t the operator of covariant differentiation with respect to the Christoffel symbols $\left\{{r \atop s\ t}\right\}$ formed with the metric tensor g_{ts}. Then Y_s being tangent to $S^n\left(\frac{1}{\sqrt{2}}\right)$ and N being unit normal to $S^n\left(\frac{1}{\sqrt{2}}\right)$, we have equations of Gauss and those of Weingarten respectively in the forms

$$\nabla_t Y_s=\sqrt{2}\, g_{ts} N,\quad \nabla_t N=-\sqrt{2}\, Y_t. \tag{1.4}$$

We now consider $S^n\left(\frac{1}{\sqrt{2}}\right)\times S^{n+1}\left(\frac{1}{\sqrt{2}}\right)$ and regard it as a submanifold of codimension 2 in a $(2n+3)$-dimensional Euclidean space E^{2n+3}. Denoting by Z the position vector of a point of $S^n\left(\frac{1}{\sqrt{2}}\right)\times S^{n+1}\left(\frac{1}{\sqrt{2}}\right)$, we have

$$Z(x^h)=\begin{pmatrix} X(x^a) \\ Y(x^r) \end{pmatrix}, \tag{1.5}$$

where here and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, n, n+1, \cdots, 2n+1\}$. Since $Z\cdot Z=X\cdot X+Y\cdot Y=1$ in E^{2n+3}, $S^n\left(\frac{1}{\sqrt{2}}\right)\times S^{n+1}\left(\frac{1}{\sqrt{2}}\right)$ is a hypersurface of $S^{2n+2}(1)$ in E^{2n+3}.

§2. $(f, g, u_{(k)}, \alpha_{(k)})$-structure on $S^n\left(\frac{1}{\sqrt{2}}\right)\times S^{n+1}\left(\frac{1}{\sqrt{2}}\right)$.

Let E^{2n+3} be a $(2n+3)$-dimensional Euclidean space with cartesian coordinates $\{y^\kappa\}$. (The indices $\kappa, \mu, \nu, \cdots$ run over the range $\{1, 2, \cdots, 2n+3\}$).

If we put

$$(\eta_\lambda)=(0, \cdots, 0, 1),$$

$$(\xi^\lambda)=\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & & & & & -1 & 0 \\ & \ddots & & 0 & \cdot\cdot\cdot & & \vdots \\ 0 & & 0 & -1 & & 0 & \vdots \\ & & 1 & 0 & & & \vdots \\ & \cdot\cdot\cdot & & 0 & \ddots & & \vdots \\ 1 & & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

then η_λ, ξ^λ and $\varphi_\mu^{\ \lambda}$ are respectively 1-form, vector and (1, 1)-type tensor in E^{2n+3} with the cartesian coordinates.

Moreover,

$$(G_{\mu\lambda})=\begin{pmatrix} 1 & & 0 \\ & 1 \ddots & \\ 0 & & 1 \end{pmatrix}$$

is a positive definite Riemannian metric in E^{2n+3}.

If we set

$$\varphi_{\mu\lambda}=\varphi_{\mu}^{\ \nu} G_{\nu\lambda},$$

then
$$\varphi_{\mu\lambda}\varphi_{\nu\rho}G^{\lambda\rho}=G_{\mu\nu}-\eta_{\mu}\eta_{\nu},$$

where
$$(G^{\lambda\rho})=(G_{\lambda\rho})^{-1}, \text{ and } G_{\lambda\mu}\xi^{\mu}=\eta_{\lambda}$$

and
$$\varphi_{\mu}^{\ \lambda}\varphi_{\lambda}^{\ \nu}=-\delta_{\mu}^{\ \nu}+\eta_{\mu}\xi^{\nu}.$$

Thus the aggregate $(\varphi_{\mu}^{\ \lambda}, \eta_{\mu}, \xi^{\lambda}, G_{\lambda\mu})$ is an almost contact metric structure in E^{2n+3} with cartesian coordinates.

Moreover, denoting $\tilde{\nabla}_{\mu}$ by the operator of covariant differentiation with respect to the Christoffel symbol $\left\{ {\lambda \atop \mu\nu} \right\}$ formed with $G_{\mu\nu}$, we find $\tilde{\nabla}_{\mu}\xi^{\nu}=0$, $\tilde{\nabla}_{\mu}\eta^{\nu}=0$ and $\tilde{\nabla}_{\mu}\varphi_{\lambda}^{\ \nu}=0$. Hence $(\varphi_{\mu}^{\ \lambda}, \eta_{\mu}, \xi^{\lambda}, G_{\lambda\mu})$ is a cosympletic structure.

In this section, we want to derive the $(f, g, u_{(k)}, \alpha_{(k)})$-structure induced on $S^{n}\left(\frac{1}{\sqrt{2}}\right)\times S^{n+1}\left(\frac{1}{\sqrt{2}}\right)$ as a submanifold of codimension 2 of E^{2n+3} with cosympletic structure.

Now, putting

$$Z_{i}=\partial_{i}Z,$$

we see that

$$Z_{b}=\begin{pmatrix} X_{b} \\ 0 \end{pmatrix}, \qquad Z_{s}=\begin{pmatrix} 0 \\ Y_{s} \end{pmatrix}.$$

and the induced Riemannian metric g_{ji} of $G_{\mu\lambda}$ has the form

$$(g_{ji})=\begin{pmatrix} g_{cb} & 0 \\ 0 & g_{ts} \end{pmatrix},$$

and hence

$$(g^{ji})=\begin{pmatrix} g^{cb} & 0 \\ 0 & g^{ts} \end{pmatrix},$$

g^{ji}, g^{cb} and g^{ts} being components of inverse matrices of (g_{ji}), (g_{cb}) and (g_{ts}) respectively.

Setting

$$C=\begin{pmatrix} -X(x^{a}) \\ -Y(x^{r}) \end{pmatrix}, \qquad D=\begin{pmatrix} -X(x^{a}) \\ Y(x^{r}) \end{pmatrix},$$

we find $Z_{i}\cdot C=0$, $Z_{i}\cdot D=0$, $C\cdot C=D\cdot D=1$ and $C\cdot D=0$, where the dot denotes the inner product reduced from $G_{\mu\lambda}$ in E^{2n+3}, and consequently that C and D are unit normals to $S^n\left(\frac{1}{\sqrt{2}}\right)\times S^{n+1}\left(\frac{1}{\sqrt{2}}\right)$.

If we denote by ∇ the induced connection on $S^n\left(\frac{1}{\sqrt{2}}\right)\times S^{n+1}\left(\frac{1}{\sqrt{2}}\right)$ from the connection $\tilde{\nabla}$ of E^{2n+3} and denote by h_{ji} and k_{ji} components of the second fundamental tensors respectively with respect to unit normals C and D, equations of Gauss are written as $\nabla_j Z_i = h_{ji}C + k_{ji}D$.

Then

$$(h_{ji}) = \begin{pmatrix} g_{cb} & 0 \\ 0 & g_{ts} \end{pmatrix}, \quad (k_{ji}) = \begin{pmatrix} g_{cb} & 0 \\ 0 & -g_{ts} \end{pmatrix}$$

and hence

$$(h_j^i) = \begin{pmatrix} \delta_b^c & 0 \\ 0 & \delta_t^s \end{pmatrix}, \quad (k_j^i) = \begin{pmatrix} \delta_c^b & 0 \\ 0 & -\delta_t^s \end{pmatrix},$$

where $h_j^i = h_{ih}g^{jh}$ and $k_j^i = k_{jh}g^{ih}$.

From these relations we have $h_{ji} = g_{ji}$, $k_h^h = -1$ and $k_j^m k_m^h = \delta_j^h$.

Also, taking account of the fact that k_j^i has the form given by the above and the Christoffel symbols $\left\{ {h \atop j\ i} \right\}$ are all zero except $\left\{ {a \atop c\ b} \right\}$ and $\left\{ {r \atop t\ s} \right\}$, denoting by l_j components of the third fundamental tensor with respect to unit normals C and D, equations of Weingarten can be written as

$$\nabla_j C = -h_j^i Z_i + l_j D,$$

$$\nabla_j D = -k_j^i Z_i - l_j C.$$

By the way, the third fundamental tensor l_j vanishes because of the definition of C and equation of Weingarten, hence $\nabla_j Z_i = g_{ji}C + k_{ji}D$, $\nabla_j C = -Z_j$ and $\nabla_j D = -k_j^i Z_i$.

Finally, we consider transforms φZ_i, φC and φD of Z_i, C and D by φ respectively:

$$\varphi Z_i = f_i^h Z_h + u_i C + v_i D, \tag{2.1}$$

$$\varphi C = -u^i Z_i + \alpha D, \tag{2.2}$$

$$\varphi D = -v^i Z_i - \alpha C, \tag{2.3}$$

where f_i^h are components of a tensor field of type (1, 1), u_i and v_i those of

1-forms and α a function of $S^n\left(\frac{1}{\sqrt{2}}\right)\times S^{n+1}\left(\frac{1}{\sqrt{2}}\right)$, u^i and v^i being respectively given by $u^i=u_j g^{ji}$ and $v^i=v_j g^{ji}$.

If we write

$$\xi=w^i Z_i+\beta C+\gamma D, \tag{2.4}$$

then from (2.1), (2.2), (2.3) and (2.4) we can easily see that $S^n\left(\frac{1}{\sqrt{2}}\right)\times S^{n+1}\left(\frac{1}{\sqrt{2}}\right)$ admits an $(f,\ g,\ u_{(k)},\ \alpha_{(k)})$-structure (See [3]), that is,

$$f_j^h f_h^i=-\delta_j^i+u_j u^i+v_j v^i+w_j w^i, \tag{2.5}$$

$$f_i^h u_h=\alpha v_i+\beta w_i,\ f_i^h u^i=-\alpha v^h-\beta w^h, \tag{2.6}$$

$$f_i^h v_h=\gamma w_i-\alpha u_i,\ f_i^h v^i=-\gamma w^h+\alpha u^h, \tag{2.7}$$

$$f_i^h w_h=-\beta u_i-\gamma v_i,\ f_i^h w^i=\beta u^h+\gamma v^h, \tag{2.8}$$

$$u^i u_i=1-\alpha^2-\beta^2,\ u^i v_i=-\beta\gamma,\ u^i w_i=\alpha\gamma, \tag{2.9}$$

$$v^i u_i=-\beta\gamma,\ v^i v_i=1-\alpha^2-\gamma^2,\ v^i w_i=-\alpha\beta, \tag{2.10}$$

$$w^i u_i=\alpha\gamma,\ w^i v_i=-\alpha\beta,\ w^i w_i=1-\beta^2-\gamma^2, \tag{2.11}$$

$$f_j^m f_i^n g_{mn}=g_{ji}-u_j u_i-v_j v_i-w_j w_i, \tag{2.12}$$

where $w_i=g_{ij}w^j$.

Moreover, putting $i=b$ in (2.1), we have $f_b^a=0$, $u_b+v_b=0$.

Also, putting $i=s$ in (2.1), we obtain $f_t^s=0$, $u_s-v_s=0$.

Thus

$$(f_i^h)=\begin{pmatrix}0 & f_s^a\\ f_b^r & 0\end{pmatrix},$$

$$(u_i)=(u_b,\ u_s),\ (u^h)=\begin{pmatrix}u^a\\ u^r\end{pmatrix},$$

where $u^a=u_b g^{ba}$, $u^r=u_s g^{sr}$ and $(v_i)=(-u_b,\ u_s)$, $(v^h)=\begin{pmatrix}-u^a\\ u^r\end{pmatrix}$

because the induced metric g_{ji} of $G_{\lambda\mu}$ has the form

$$\begin{pmatrix} g_{ba} & 0 \\ 0 & g_{st} \end{pmatrix}$$

Thus $k_j^{\ h}u^j=-v^h$, $k_j^{\ h}v^j=-u^h$, and moreover, $k_{jm}f_i^{\ m}-k_{im}f_j^{\ m}=0$.

On the other hand, applying the operator ∇ of covariant differentiation to (2.1), (2.2), (2.3) and (2.4) and taking account of $\tilde{\nabla}\varphi=0$, $\tilde{\nabla}\eta=0$ and $\tilde{\nabla}\xi=0$, we get

$$\nabla_j f_i^{\ h}=-g_{ji}u^h-k_{ji}v^h+\delta_j^{\ h}u_i+k_j^{\ h}v_i,$$

$$\nabla_j u_i=f_j^{\ i}-\alpha k_{ji},$$

$$\nabla_j v_i=-k_{jh}f_i^{\ h}+\alpha g_{ji},$$

$$\nabla_j w_i=\beta g_{ji}+\gamma k_{ji},$$

$$\nabla_j\alpha=k_{ji}u^i-v_j=-2v_j,$$

$$\nabla_j\beta=-w_j,$$

$$\nabla_j\gamma=-k_j^{\ i}w_i,$$

$$\nabla_j\nabla_i u_h=-g_{ji}u_h+g_{jh}u_i-k_{ji}v_h+k_{jh}v_i+2k_{ih}v_j.$$

Moreover,

$$S_{ji}^{\ \ h}=-(\nabla_j f_i^{\ m})f_m^{\ h}+(\nabla_i f_j^{\ m})f_m^{\ h}+f_j^{\ m}\nabla_m f_i^{\ h}-f_i^{\ m}\nabla_m f_j^{\ h}$$

$$+(\nabla_j u_i-\nabla_i u_j)u^h+(\nabla_j v_i-\nabla_i v_j)v^h+(\nabla_j w_i-\nabla_i w_j)w^h$$

$$=-2(k_j^{\ m}f_m^{\ h}v_i-k_i^{\ m}f_m^{\ h}v_j)$$

$$=2v_j(\nabla_i v^h-\alpha\delta_i^{\ h})-2v_i(\nabla_j v^h-\alpha\delta_j^{\ h}).$$

§3. A characterization of $S^n\times S^{n+1}$.

In this section, we study complete Riemannian manifolds admitting (f, g, $u_{(k)}$, $\alpha_{(k)}$)-structures which satisfies some of differential equations obtained in the last part of §2.

We first prove

THEOREM 1. *Let M be a complete and connected $n(>2)$-dimensional Riemannian manifold M with metric tensor g_{ji}, and assume that there exist in M a symmetric tensor field k_{ji} and a skew-symmetric tensor field f_{ji} which satisfy*

(3.1) $\qquad trace(k_j^{\ i})=constant,$

(3.2) $$-(n-2)\sqrt{A}\leq trace(k_j^{\ i})\leq(n-2)\sqrt{A},$$

(3.3) $$\nabla_k k_{ji}-\nabla_j k_{ki}=0,$$

(3.4) $$k_{jm}k_i^{\ m}=Ag_{ji},$$

A being a differentiable function, and there exists a non-trivial differentiable function α such that

(3.5) $$\nabla_j\nabla_i\alpha=2k_{jm}f_i^{\ m}-2\alpha g_{ji},$$

where $$k_j^{\ i}=k_{jm}g^{im} \text{ and } f_j^{\ i}=f_{jm}g^{im}.$$

Then, M is globally isometric to $S^n\left(\frac{1}{\sqrt{2}}\right)$ *or* $S^p\left(\frac{1}{\sqrt{2}}\right)\times S^{n-p}\left(\frac{1}{\sqrt{2}}\right)$ *or* $\left[S^p\left(\frac{1}{\sqrt{2}}\right)\times S^{n-p}\left(\frac{1}{\sqrt{2}}\right)\right]^*$, $(2\leq p\leq n-2)$, $S^p\left(\frac{1}{\sqrt{2}}\right)$ *being p-dimensional sphere with radius* $\frac{1}{\sqrt{2}}$.

PROOF. Differentiating (3.4) covariantly, we find

(3.6) $$(\nabla_k k_{jm})k_i^{\ m}+k_{jm}(\nabla_k k_i^{\ m})=(\nabla_k A)g_{ji},$$

from which, contracting j and i and using (3.3), $2k_{ji}(\nabla_k k^{ji})=(2n+1)\nabla_k A$.

If we contract again k and i in (3.6) and use (3.1) and (3.2), (3.6) can be written as

$$k_{im}(\nabla_j k^{im})=\nabla_j A.$$

From the last two equations we have A=constant.

If $A=0$, then we have from (3.4), $k_{ji}=0$.

Thus (3.5) becomes $\nabla_j\nabla_i\alpha=-(\sqrt{2})^2\alpha g_{ji}$.

Since M is complete, by the theorem of Obata, M is isometric to a sphere $S^{2n+1}\left(\frac{1}{\sqrt{2}}\right)$.

Since A is a constant, we consider only $A\neq 0$.

We have from (3.5),

(3.7) $$k_{jm}f_i^{\ m}-k_{im}f_j^{\ m}=0.$$

Putting

(3.8) $$P_i^{\ h}=\frac{1}{2}\left(\delta_i^{\ h}+\frac{1}{\sqrt{A}}k_i^{\ h}\right),$$

we have from (3.4), (3.5) and (3.7)

(3.9) $$P_l^{\ i}\nabla_j\nabla_i\alpha=k_{jm}f_l^{\ m}+\sqrt{A}f_{jl}-\frac{\alpha}{\sqrt{A}}k_{jl}-\alpha g_{jl},$$

from which, using (3.4) and (3.6),

(3.10) $$P_k^{\ j}P_l^{\ i}\nabla_j\nabla_i\alpha=-2\alpha P_{kl}$$

If we put $Q_i^{\ h}=\delta_i^{\ h}-P_i^{\ h}$, then we can see that $P_i^{\ l}P_l^{\ h}=P_i^{\ h}$, $P_i^{\ l}Q_l^{\ h}=0$, $Q_i^{\ l}Q_l^{\ h}=Q_i^{\ h}$ by virtue of (3.8).

Using (3.3) and A=constant, we find

$$\nabla_k P_i^{\ h}-\nabla_i P_k^{\ h}=0,$$

from which, $\nabla_k P_i^{\ h}=0$ (See Lemma 1.1 in [2]).

Thus $P_i^{\ h}$ and $Q_i^{\ h}$ are two conplementary almost product structures such that $\nabla_k P_i^{\ h}=0$.

Moreover, from (3.5) and (3.9), we get

$$Q_l^{\ i}\nabla_j\nabla_i\alpha=(\delta_l^{\ i}-P_l^{\ i})\nabla_j\nabla_i\alpha$$

$$=k_{jm}f_l^{\ m}-\alpha g_{jl}-\sqrt{A}f_{jl}+\frac{\alpha}{\sqrt{A}}k_{jl},$$

from which, using (3.8) and (3.10),

$$Q_k^{\ j}Q_l^{\ i}\nabla_j\nabla_i\alpha=-2\alpha Q_{kl}.$$

On the other hand, from (3.2) and (3.8), we find $2\leq\text{rank }(P_i^{\ h})\leq n-2$. Therefore, the assumptions of Theorem A are all satisfied and consequently the conclusions of Theorem A are valid.

We next prove

THEOREM 2. *Assume that a complete and connected $(2n+1)$-dimensional differentiable manifold M admits an $(f, g, u_{(k)}, \alpha_{(k)})$-structure such that $\alpha^2+\beta^2+\gamma^2\neq 1$, $\alpha\neq 0$, $\beta\neq 0$ and $\gamma\neq 0$ almost everywhere, and*

(3.11) $$\nabla_j\alpha=-2v_i,\quad \nabla_j\beta=-w_j.$$

If there exists a tensor field k_{ji} of type (0, 2) which satisfies

(3.12) $$\nabla_j u_i=f_{ji}-\alpha k_{ji},$$

and

(3.13) $$\nabla_k\nabla_j u_i=-g_{kj}u_i+g_{ki}u_j-k_{kj}v_i+k_{ki}v_j+2k_{ji}v_k,$$

where ∇ denotes the Levi-Civita connection induced from the Riemannian metric tensor g_{ji}. Then M is isometric to $S^{n+1}\left(\frac{1}{\sqrt{2}}\right)\times S^n\left(\frac{1}{\sqrt{2}}\right)$ or $\left[S^{n+1}\left(\frac{1}{\sqrt{2}}\right)\times S^n\left(\frac{1}{\sqrt{2}}\right)\right]^*$.

PROOF. We have from (3.12)

$$\nabla_j u_i + \nabla_i u_j = -2\alpha k_{ji},$$

from which, differentiating covariantly and substituting (3.11) and (3.13), $\alpha \nabla_k k_{ji} = 0$. Since α is almost everywhere non-zero in M, we have $\nabla_k k_{ji} = 0$.

If we differentiate covariantly and take account of (3.11), (3.13) and $\nabla_k k_{ji} = 0$, we see that

$$\nabla_k f_{ji} = -g_{kj} u_i + g_{ki} u_j - k_{kj} v_i + k_{ki} v_j. \tag{3.14}$$

On the other hand, we have from (3.12)

$$\nabla_j u_i - \nabla_i u_j = 2 f_{ji},$$

from which, transvecting u^j and substituting (2.6) and (2.9)

$$u^j \nabla_j u_i = \frac{1}{2} \nabla_i (1 - \alpha^2 - \beta^2) - 2(\alpha v_i + \beta w_i),$$

from which, using (3.11)

$$u^j \nabla_j u_i = -\beta w_i. \tag{3.15}$$

Transvecting (3.12) with u^j and using (2.6) and (3.15), we obtain

$$k_{ji} u^i = -v_j \tag{3.16}$$

because α is almost everywhere non-zero.

Differentiating (3.16) covariantly and taking account of $\nabla_k k_{ji} = 0$ and (3.12), we have

$$-\nabla_k v_j = k_{jm} f_k{}^m - \alpha k_{jm} k_k{}^m. \tag{3.17}$$

From the first equation of (3.11) we see that $\nabla_j v_k - \nabla_k v_j = 0$. Thus (3.17) implies that

$$k_{jm} f_k{}^m - k_{km} f_j{}^m = 0. \tag{3.18}$$

Transvecting u^k to (3.18) and using (2.6), (2.7) and (3.16), we get

$$\alpha k_{jm} v^m = -\beta k_{jm} w^m + \gamma w_j - \alpha u_j. \tag{3.19}$$

Transvecting again (3.18) with f^{jk} and making use of (2.5) and the skew-symmetry of f^{jk}, we find

$$k_m{}^m = k_{ji} u^j u^i + k_{ji} v^j v^i + k_{ji} w^j w^i. \tag{3.20}$$

Differentiating (3.18) covariantly and substituting (3.14), we obtain

$$k_{jm}(-g_{ki} u^m + \delta_k{}^m u_i - k_{ki} v^m + k_k{}^m v_i)$$
$$= k_{im}(-g_{kj} u^m + \delta_k{}^m u_j - k_{kj} v^m + k_k{}^m v_j)$$

by virtue of $\nabla_k k_{ji} = 0$, or, using (3.16)

$$g_{ki}v_j+k_{jk}u_i-(k_{jm}v^m)\,k_{ki}+(k_{jm}k_k{}^m)v_i$$
$$=g_{kj}v_i+k_{ik}u_j-(k_{im}v^m)\,k_{kj}+(k_{im}k_k{}^m)v_j. \tag{3.21}$$

Differentiating the second equation of (2.9) covariantly, we find

$$(\nabla^j u^i)v^i+u^i(\nabla_j v_i)=-(\nabla_j\beta)\gamma-\beta\nabla_j\gamma,$$

from which, substituting (3.11), (3.12) and (3.17),

$$\beta\nabla_j\gamma=\gamma w_j-(f_j^{\ i}-\alpha k_j^{\ i})v_i+u^i(k_{im}f_j^{\ m}-\alpha k_{im}k_j^{\ m}),$$

or, using (2.7) and (3.16),

$$\beta\nabla_j\gamma=2\alpha k_{jm}v^m+2\alpha u_j-\gamma w_j. \tag{3.22}$$

Differentiating the first equation of (2.7) covariantly, we find

$$(\nabla_j f_i^{\ h})v_h+f_i^{\ h}(\nabla_j v_h)=(\nabla_j\gamma)w_i+\gamma\nabla_j w_i-(\nabla_j\alpha)u_i-\alpha\nabla_j u_i,$$

from which, taking skew-symmetric parts,

$$(\nabla_j f_i^{\ h}-\nabla_i f_j^{\ h})v_h+f_i^{\ h}(\nabla_j v_h)-f_j^{\ h}(\nabla_i v_h)$$
$$=(\nabla_j\gamma)w_i-(\nabla_i\gamma)w_j-(\nabla_j\alpha)u_i+(\nabla_i\alpha)u_j-\alpha(\nabla_j u_i-\nabla_i u_j),$$

or, using (3.11), (3.12), (3.14) and (3.17),

$$-(v_ju_i-v_iu_j)+(k_{jm}v^m)v_i-(k_{im}v^m)v_j+2\alpha f_i^{\ h}k_{hm}k_j^{\ m}$$
$$=(\nabla_j\gamma)w_i-(\nabla_i\gamma)w_j-2\alpha f_{ji}. \tag{3.23}$$

Transvecting (3.23) with u^j and making use of (2.6), (2.7), (2.9), (2.10) and (3.16), we have

$$(\gamma^2-\beta^2)v_i+\beta\gamma(k_i^{\ m}v_m)+2\alpha\gamma\,k_{im}w^m$$
$$=(u^m\nabla_m\gamma)w_i-\alpha\gamma\nabla_i\gamma-\beta\gamma\,u_i+2\alpha\beta\,w_i, \tag{3.24}$$

from which, using (3.19),

$$\beta(\gamma^2-\beta^2)v_i+\beta^2\gamma(k_i^{\ m}v_m)+2\alpha\gamma(-\alpha k_{im}v^m+\gamma w_i-\alpha u_i)$$
$$=\beta(u^m\nabla_m\gamma)w_i-\alpha\beta\gamma\nabla_i\gamma-\beta^2\gamma u_i+2\alpha\beta^2 w_i.$$

Since $\beta u^m\nabla_m\gamma=-2\alpha\beta^2+\alpha\gamma^2$, from (3.16) and (3.22), the above equation becomes $\beta^2\gamma(k_{im}v^m)=-\beta^2\gamma u_i+\beta(\beta^2-\gamma^2)v_i$, and consequently

$$k_{im}v^m=-u_i+((\beta^2-\gamma^2)/\beta\gamma)v_i. \tag{3.25}$$

Substituting (3.25) into (3.21), (3.21) becomes

(3.26) $v_j\{g_{ki}-((\beta^2-\gamma^2)/\beta\gamma)k_{ki}-k_{km}k_i^m\}=v_i\{g_{kj}-((\beta^2-\gamma^2)/\beta\gamma)k_{kj}-k_{km}k_j^m\}$.

Using (3.25), we also find

$$v^m[g_{km}-((\beta^2-\gamma^2)/\beta\gamma)k_{km}-k_{kj}k_m^j]=-2((\beta^2-\gamma^2)/\beta\gamma)\{-u_k+((\beta^2-\gamma^2)/\beta\gamma)v_k\}.$$

Transvecting (3.26) with v^j and using the above equation, we get

$$(3.27)\qquad 0=(1-\alpha^2-\gamma^2)\{g_{ki}-((\beta^2-\gamma^2)/\beta\gamma)k_{ki}-k_{km}k_i^m\}$$
$$+2((\beta^2-\gamma^2)/\beta\gamma)\{-v_iu_k+((\beta^2-\gamma^2)/\beta\gamma)v_iv_k\},$$

from which $0=((\beta^2-\gamma^2)/\beta\gamma)(v_ku_i-v_iu_k)$, and consequently

$$(3.28)\qquad \beta^2-\gamma^2=0.$$

Thus, using (3.27) and (3.28),

$$(3.29)\qquad k_{jm}k_i^m=g_{ji}.$$

Also, from (3.19), (3.25) and (3.28), we have

$$(3.30)\qquad k_{jm}v^m=-u_j,\quad k_{jm}w^m=\frac{\gamma}{\beta}w_j.$$

Moreover, from (3.11) and (3.17),

$$(3.31)\qquad \nabla_k\nabla_j\alpha=2k_{km}f_j^m-2\alpha g_{kj},$$

and, using (2.9), (2.11), (3.16), (3.20) and (3.30),

$$(3.32)\qquad k_m^m=\frac{\gamma}{\beta}=\pm 1$$

by virtue of (3.28).

Since the manifold is connected, $k_m^m=1$ or $k_m^m=-1$ on the whole space. Thus the equations (3.29), (3.31), (3.32), $\nabla_k k_{ji}=0$ and Theorem 1 prove the theorem.

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