Kyungpook Math. J.
Volume 13, Number 2
December, 1973

ON THE CONSTRUCTION OF MULTIPLE PARAMETER EXTENSORS OF HIGHER ORDER

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The generalized geometric object 'Extensor' was originally introduced by Craig [1][1]) for one parameter, and subsequently, he successfully generalized the idea for multiple parameter extensors in [4, 5]. Lateron, the extensors were extensively studied by him with application to geometry, mechanics and the calculus of variations problem. (See bibl. [6] and papers [7, 8, 9]). A long ago, in his very interesting paper [3], Craig constructed the extensors of higher order as intrinsic derivatives of higher order tensors and led the way for their generalisations.

The prime purpose of the present paper is to develop the theory in which we shall construct multiple parameter extensors from tensors by differentiation.

Notations to be employed in the present paper are the same as those used by Craig in his works listed at the end of the paper. In this paper, we shall employ at most two coordinate systems x and $\bar{x}$, but we shall use only one root letter x for both systems, and distinguish them by restricting the choise of indicial letters. Letters at the beginning of the alphabet $a, b, c, d, \cdots; \alpha, \beta, \gamma, \delta, \cdots$ are to be associated with x coordinate system, while the others with $\bar{x}$ system. The Greek indices will be used both to denote integers 0 to M and to denote matrices throughout this paper. If the variables x^a are made functions of two (or more) parameters, say u_1 and u_2, so that we have $x^a = x^a(u_1, u_2)$, then the mixed order partial derivatives with respect to parameters such as $\dfrac{\partial^{\alpha_1+\alpha_2} F}{\partial u_1^{\alpha_1} \partial u_2^{2\alpha_2}}$ may be denoted by $F^{(\alpha_1, \alpha_2)}$. The symbol $X^{\rho r}_{\alpha a}$ for $\rho = (\rho_1, \rho_2)$ $\alpha = (\alpha_1, \alpha_2)$ denotes $\dfrac{\partial x^r(\rho_1, \rho_2)}{\partial x^a(\alpha_1, \alpha_2)}$. Repeated lower case Latin indices indicate summations from 1 to N, while lower case Greek indices, unless contrary is indicated, from 0 to M in the case of one

1) Numbers in brackets refer to the references at the end of the paper.

parameter, and over the set of matrices (α_1, α_2) with α_1 ranging from 0 to M_1, and α_2 from 0 to M_2. Capital letters do not generate the sums. Now, we shall contemplate on some of the fundamental notions.

Let F be a scalar or vector point function, then

$$F^{(M)}_{(\alpha)a}=\binom{M}{A}F^{(M-A)}_a, \qquad A=\alpha,$$

where the symbol $\binom{M}{A}$ denotes the binomial coefficient, consequently, we have

$$(1) \qquad X^{\rho r}_{\alpha a}=\binom{P}{A}X^{r(P-A)}_a; \; P=\rho,\; A=\alpha, \text{ if } P>A,$$
$$=X^r_a, \qquad \text{if } P=A,$$
$$=0, \qquad \text{if } P<A.$$

The expansion of the derivation $(FG)^M$, that is, the M times differentiation of a product of two functions F and G with application of the Leibnitz rule of differentiation, gives

$$(2) \qquad (FG)^M=\sum_{\alpha=0}^{M}\binom{M}{\alpha}F^{(\alpha)}G^{(M-\alpha)}.$$

The symbols $U_{\alpha a}$ and $V^{\alpha a}$ as used in this paper are defined as follows:

$$(3) \qquad V^{\alpha a}=V^{(\alpha_1,\alpha_2)a}=\frac{\partial^{\alpha_1+\alpha_2}V^a}{\partial u_1^{\alpha_1}\partial u_2^{\alpha_2}},$$

$$(4) \qquad U_{\alpha a}=U_{(\alpha_1,\alpha_2)a}=\binom{M_1}{A_1}\binom{M_2}{A_2}U^{(M_1-A_1,M_2-A_2)}, \quad A_1=\alpha_1,\; A_2=\alpha_2,$$

range: $\alpha_1 : 0$ to M_1, $\alpha_2 : 0$ to M_2.

Furthermore, before going to our actual course of discussion, we shall review the generalized binomial coefficients, which appeared in the works of Craig [3]. Some typical special cases among them are defined as follows:

$$\binom{M}{A,\; B,\; \Gamma}=\frac{M!}{A!\,B!\,\Gamma!\,(M-A-B-\Gamma)!}, \qquad \text{if } M\geq A+B+\Gamma,$$
$$=0, \qquad \text{if } M<A+B+\Gamma,$$

$$\begin{bmatrix}A,\; B,\; \Gamma\\ M\end{bmatrix}=\binom{M}{M-A,\; M-B,\; M-\Gamma}\binom{M}{A}^{-1}\binom{M}{B}^{-1}\binom{M}{\Gamma}^{-1}, \quad \text{if } A+B+\Gamma\geq 2M,$$
$$=0, \qquad \text{if } A+B+\Gamma<2M,$$

$$\begin{Bmatrix}A,\; B\\ M,\; \Gamma,\; \Delta\end{Bmatrix}=\begin{bmatrix}A,\; B\\ M\end{bmatrix}\binom{A+B-M}{\Gamma,\; \Delta}=\frac{A!\,B!}{M!\,\Gamma!\,\Delta!(A+B-M-\Gamma-\Delta)!}.$$

Now, first let us suppose that the quantities $E_{\alpha_1 a}$, defined by

$$E_{\alpha_1 a}=\binom{M_1}{A_1}T_a^{(M_1-A_1)}, \quad A_1=\alpha_1,\ M_1>A_1,$$

derived from the tensor T_a are functions of u_1 and u_2 and are extensor components relative to u_1, so that

$$E_{\rho_1 r}=E_{\alpha_1 a}X^{\alpha_1 a}_{\rho_1 r},$$

$$X^{\alpha_1 a}_{\rho_1 r}=\binom{A_1}{P_1}X_r^{a[(A_1,0)-(P_1,0)]}=\binom{A_1}{P_1}\partial^{A_1-P_1}X^a_r/\partial u_1^{A_1-P_1},$$

$$A_1=\alpha_1,\ P_1=\rho_1,\ \text{range: } \alpha_1,\rho_1: 0 \text{ to } M_1.$$

Similarly, let $V^{(\rho_1,0)r}$ be an extensor derived from the tensor V^r, we then have

$$(V^{(\rho_1,0)r}E_{\rho_1 r})^{(0,M_2)}=(V^{(\alpha_1,0)a}E_{\alpha_1 a})^{(0,M_2)}=\sum_{\alpha_2=0}^{M_2}\binom{M_2}{\alpha_2}V^{(\alpha_1,\alpha_2)a}E_{\alpha_1 a}^{\cdots[0,(M_2-\alpha_2)]}$$

If we define $E_{(\alpha_1,\alpha_2)a}$ by

$$E_{(\alpha_1,\alpha_2)a}=\binom{M_2}{\alpha_2}E_{\alpha_1 a}^{\cdots[0,(M_2-\alpha_2)]}, \quad \text{range: } \alpha_2: 0 \text{ to } M_2,$$

then by quotient law $E_{(\alpha_1,\alpha_2)a}$ is a two parameter extensor. Hence, we have the

THEOREM 1. *Let T_a be an absolute tensor of the type* (0, 1, 0), *then $E_{(\alpha_1,\alpha_2)a}$ is a two parameter extensor of the type* (0, 1, 0, 0, 0) *derived from the tensor T_a.*

REMARK 1.1. On making use of the mathematical induction, this theorem of course can be extended to higher type of multiple parameter extensor and we can get the $E_{(\alpha_1,\alpha_2,\cdots,\alpha_k)a}$, an extensor of k-parameter.

Next, we suppose that the quantities $E^{\alpha_1 a}$ defined by $E^{\alpha_1 a}=T^{(\alpha_1)a}$ derived from the tensor T^a are functions of u_1 and u_2, and are components of an excontravariant extensor relative to u_1, so that we have

$$E^{\alpha_1 a}=E^{\rho_1 r}X^{\alpha_1 a}_{\rho_1 r},$$

$$X^{\alpha_1 a}_{\rho_1 r}=\binom{A_1}{P_1}X_r^{a[(A_1,0)-(P_1,0)]}=\binom{A_1}{P_1}\partial^{A_1-P_1}X^a_r/\partial u_1^{A_1-P_1},$$

$$A_1=\alpha_1,\ P_1=\rho_1,\ \text{range: } \alpha_1,\rho_1: 0 \text{ to } M_1.$$

Similarly, let $V_{(\alpha_1,0)a}$ be an extensor derived from the tensor V_a, then by expanding $(V_{(\alpha_1,0)a}E^{\alpha_1 a})^{(0,M_2)}$ with application of the Leibnitz rule of differentiation (2), we obtain

$$(V_{(\alpha_1,0)a}E^{\alpha_1 a})^{(0,M_2)}=\sum_{\alpha_2=0}^{M_2}\binom{M_2}{\alpha_2}V^{\cdots(\alpha_2)}_{(\alpha_1,0)a}\,E^{\alpha_1 a[0,(M_2-\alpha_2)]}.$$

Replacing the dummy index α_2 with $M_2-\bar{\alpha}_2$, droping the bars and using the definition

$$(5)\qquad V_{(\alpha_1,\alpha_2)a}=\binom{M_2}{A_2}V^{\cdots(M_2-A_2)}_{(\alpha_1,0)a},\quad A_2=\alpha_2,\ \text{range: } \alpha_2\text{: } 0 \text{ to } M_2,$$

the R.H.S. of above relation becomes $V_{(\alpha_1,\alpha_2)a}E^{(\alpha_1,\alpha_2)a}$. From which by quotient law, we conclude that $E^{(\alpha_1,\alpha_2)a}$ is a two parameter extensor of excontravariant order one.

THEOREM 2. *Let T^a be a contravariant tensor of order one, then the quantity $E^{(\alpha_1,\alpha_2)a}$ is a two parameter extensor of excontravariant order one, derived from the tensor T^a.*

REMARK 2.1. If we extend this theorem by making use of the mathematical induction, we can easily get a k-parameter extensor $E^{(\alpha_1,\alpha_2,\cdots,\alpha_k)a}$.

Now, in the forthcoming part of this paper, we devote ourselves in developing the construction of the multiple parameter extensors of higher order.

THEOREM 3. *If T^{abc} is a tensor of the type* (3, 0, 0) *and the necessary derivatives exist, then the quantities $E^{(\alpha_1,\alpha_2)a.(\beta_1,\beta_2)b.(\gamma_1,\gamma_2)c}$ derived from the tensor T^{abc}, are the components of a two parameter extensor of the type* (3, 0, 0, 0, 0).

PROOF. Let T^{abc} be a tensor of the type (3, 0, 0), and the quantities $E^{\alpha_1 a.\beta_1 b.\gamma_1 c}$ defined by

$$E^{\alpha_1 a.\beta_1 b.\gamma_1 c}=\begin{bmatrix}A_1,\ B_1,\ \Gamma_1\\ M_1\end{bmatrix}T^{abc(A_1+B_1+\Gamma_1-2M_1)},\quad A_1=\alpha_1,\ B_1=\beta_1,\ \Gamma_1=\gamma_1.$$

range: α_1, β_1, γ_1: 0 to M_1,

are the components of an extensor of the type (3, 0, 0, 0, 0) derived from the tensor T^{abc} (Craig [3]). We now assume that the quantities $E^{\alpha_1 a.\beta_1 b.\gamma_1 c}$ are functions of u_1 and u_2 and are extensor components relative to u_1, so that we have

$$E^{\alpha_1 a.\beta_1 b.\gamma_1 c}=E^{\rho_1 r.\sigma_1 s.\tau_1 t}X^{\alpha_1 a}_{\rho_1 r}X^{\beta_1 b}_{\sigma_1 s}X^{\gamma_1 c}_{\tau_1 t}.$$

Similarly, let $U_{(\alpha_1,0)a}$, $V_{(\beta_1,0)b}$, $W_{(\gamma_1,0)c}$ are the extensors derived from the tensors U_a, V_b and W_c respectively. We then can construct the invariant $(U_{(\alpha_1,0)a}\ V_{(\beta_1,0)b}\ W_{(\gamma_1,0)c}\ E^{(\alpha_1 a.\beta_1 b.\gamma_1 c)})$ for arbitrary choice of the excovariant extensors U, V, W. Differentiating this invariant over the set of matrix differentiator $(0, M_2)$ by application of the Leibnitz rule of differentiation of a multinomial, we get

$$(U_{(\alpha_1, 0)a}\, V_{(\beta_1, 0)b}\, W_{(\gamma_1, 0)c} E^{\alpha_1 a.\,\beta_1 b.\,\gamma_1 c})^{(0, M_2)}$$

$$=\sum \binom{M_2}{\alpha_2, \beta_2, \gamma_2} U^{\cdots(\alpha_2)}_{(\alpha_1, 0)a}\, V^{\cdots(\beta_2)}_{(\beta_1, 0)b}\, W^{\cdots(\gamma_2)}_{(\gamma_1, 0)c}\, E^{\alpha_1 a.\,\beta_1 b.\,\gamma_1 c[0,\,(M_2-\alpha_2-\beta_2-\gamma_2)]}$$

range: α_2, β_2, γ_2: 0 to M_2.

We now, replace the dummy indices α_2, β_2, γ_2 with $M_2-\bar{\alpha}_2$, $M_2-\bar{\beta}_2$, $M_2-\bar{\gamma}_2$, drop the bars, express $U_{(\alpha_1, 0)a}$ sup$(M_2-\alpha_2)$ interms of $U_{(\alpha_1, \alpha_2)a}$ by using the relation (5), and if we define

$$E^{(\alpha_1, \alpha_2)a.\,(\beta_1, \beta_2)b.\,(\gamma_1, \gamma_2)c}=\begin{bmatrix} A_2, B, \Gamma_2 \\ M_2 \end{bmatrix} E^{\alpha_1 a.\,\beta_1 b.\,\gamma_1 c[0,\,(A_2+B_2+\Gamma_2-2M_2)]},$$

$$A_2=\alpha_2,\ B_2=\beta_2,\ \Gamma_2=\gamma_2,$$

the result is obtained into the form

$$U_{(\alpha_1, \alpha_2)a}\, V_{(\beta_1, \beta_2)b}\, W_{(\gamma_1, \gamma_2)c}\, E^{(\alpha_1, \alpha_2)a.\,(\beta_1, \beta_2)b.\,(\gamma_1, \gamma_2)c}$$

From which by quotient law, we conclude that $E^{(\alpha_1, \alpha_2)a.\,(\beta_1, \beta_2)b.\,(\gamma_1, \gamma_2)c}$ is a two parameter extensor. Hence, the theorem follows.

REMARK 3.1. Similarly constructing the invariant $(U_{(\alpha_1, \alpha_2, 0)a}\, V_{(\beta_1, \beta_2, 0)b}\, W_{(\gamma_1, \gamma_2, 0)c}\, E^{(\alpha_1, \alpha_2)a\cdot(\beta_1, \beta_2)b\cdot(\gamma_1, \gamma_2)c})$ and so on, differentiating it over the set of matrix differentiators $(0, 0, M_3)$, $(0, 0, 0, M_4)$, and so on successively, and following the same type of procedure as in the proof, this theorem can of course be extended to the higher type of multiple parameter extensor and we can get a k-parameter extensor $E^{(\alpha_1, \alpha_2, \cdots, \alpha_k)a.\,(\beta_1, \beta_2, \cdots, \beta_k)b.\,(\gamma_1, \gamma_2, \cdots, \gamma_k)c}$ of the type (3, 0, 0, 0, 0).

REMARK 3.2. In the proof of the preceding theorem, if we operate on the quantities $E^{\alpha_1 a.\,\beta_1 b.\,\gamma_1 c\cdots}$ defined by

$$E^{\alpha_1 a.\,\beta_1 b.\,\gamma_1 c\cdots\cdots}=\begin{bmatrix} A, B_1, \Gamma_1\cdots \\ M_1 \end{bmatrix} T^{abc\cdots}(A_1+B_1+\Gamma_1+\cdots-(q-1)M_1),$$

(Craig [3] p. 335) which are one parameter extensor components of unspecified qth order derived from the tensor $T^{abc\cdots}$ of the type $(q, 0, 0)$, by constructing the invariant for arbitrary choice of the extensors $U_{(\alpha_1, 0)a}$, $V_{(\beta_1, 0)b}$, $W_{(\gamma_1, 0)c}$ and so on, we can easily get the quantities $E^{(\alpha_1, \alpha_2)a.\,(\beta_1, \beta_2)b.\,(\gamma_1, \gamma_2)c\cdots\cdots}$ defined by

$$E^{(\alpha_1, \alpha_2)a.\,(\beta_1, \beta_2)b.\,(\gamma_1, \gamma_2)c\cdots\cdots}=\begin{bmatrix} A_2, B_2, \Gamma_2, \cdots \\ M_2 \end{bmatrix} E^{\alpha_1 a.\,\beta_1 b.\,\gamma_1 c\cdots[0,\,(A_2+B_2+\Gamma_2+\cdots-(q-1)M_2)]},$$

which are two parameter extensor components of unspecified order q.

THEOREM 4. *If T_{cd} is a tensor of the type* (0, 2, 0), *then the quantities $E_{(\gamma_1,\gamma_2)c.(\delta_1,\delta_2)d}$ derived from the tensor T_{cd}, are the components of a two parameter extensor of the type* (0, 2, 0, 0, 0).

PROOF. Let T_{cd} be a covariant tensor of order two, then the quantities $E_{\gamma_1 c.\delta_1 d}$ defined by

$$E_{\gamma_1 c.\delta_1 d}=\binom{M_1}{\Gamma_1,\,\Delta_1}T_{cd}{}^{(M_1-\Gamma_1-\Delta_1)},\quad \Gamma_1=\gamma_1,\ \Delta_1=\delta_1,$$

range: γ_1, δ_1: 0 to M_1,

are the one parameter components of an extensor of the type (0, 2, 0, 0, 0) (Craig [3]). We now suppose that the quantities $E_{\gamma_1 c.\delta_1 d}$ are the functions of u_1 and u_2 and are extensor components relative to u_1 so that we have

$$E_{\gamma_1 c.\delta_1 d}=E_{\tau_1 t.\nu_1 u}X^{\tau_1 t}_{\gamma_1 c}X^{\nu_1 u}_{\delta_1 d}.$$

Similarly, let $U^{(\gamma,0)c}$, $V^{(\delta_1,0)d}$ be the extensors derived from the tensors U^c, V^d. We can then construct the invariant $(U^{(\gamma_1,0)c},\ V^{(\delta_1,0)d},\ E_{\gamma_1 c.\delta_1 d})$ for arbitrary choice of extensors $U^{(\gamma_1,0)c}$, $V^{(\delta_1,0)d}$. Differentiating this invariant over the set of matrix differentiator $(0, M_2)$ with the application of the Leibnitz rule of differentiation of a multinomial, we have

$$(U^{(\gamma_1,0)c}\quad V^{(\delta_1,0)d}E_{\gamma_1 c.\delta_1 d})^{(0,M_2)}=\sum\binom{M_2}{\gamma_2,\,\delta_2}U^{(\gamma_1,\gamma_2)c}V^{(\delta_1,\delta_2)d}E_{\gamma_1 c.\delta_1 d}^{\cdots\cdots[0,(M_2-\gamma_2-\delta_2)]}$$

range: γ_2, δ_2: 0 to M_2.

If we define the quantities $E_{(\gamma_1,\gamma_2)c.(\delta_1,\delta_2)d}$ by

$$E_{(\gamma_1,\gamma_2)c.(\delta_1,\delta_2)d}=\binom{M_2}{\Gamma_2,\,\Delta_2}E_{\gamma_1 c.\delta_1 d}^{\cdots\cdots[0,(M_2-\Gamma_2-\Delta_2)]},\quad \Gamma_2=\gamma_2,\ \Delta_2=\delta_2,$$

then by quotient law, $E_{(\gamma_1,\gamma_2)c.(\delta_1,\delta_2)d}$ is a two parameter extensor of excovariant order two, and the theorem follows.

REMARK 4.1. This theorem can obviously be extended to the higher type of multiple parameter extensor by constructing the invariant $(U^{(\gamma_1,\gamma_2,0)c}\quad V^{(\delta_1,\delta_2,0)d}\ E_{(\gamma_1,\gamma_2)c.(\delta_1,\delta_2)d})$, and so on, differentiating it over the set of matrix differentiators $(0,\ 0,\ M_3)$, $(0,\ 0,\ 0,\ M_4)$ and so on, successively, and proceeding as before we can get with ease the k-parameter extensor $E_{(\gamma_1,\gamma_2,\cdots,\gamma_k)c.(\delta_1,\delta_2,\cdots,\delta_k)d}$ of the type (0, 2, 0, 0, 0).

REMARK 4.2. In the proof of the preceding theorem, if we operate on the invariant $(U^{(\gamma_1,0)c}\ V^{(\delta_1,0)d}\cdots E_{\gamma_1 c.\,\delta_1 d\cdots})^{(0,M_2)}$, which is formed with respect to the extensor quantity $E_{\gamma_1 c.\,\delta_1 d\cdots}$ defined by

$$E_{\gamma_1 c.\,\delta_1 d\cdots}=\binom{M_1}{\Gamma_1,\ \Delta_1,\ \cdots}T_{cd}{}^{(M_1-\Gamma_1-\Delta_1)},$$

and for the arbitrary choice of extensors $U^{(\gamma_1,0)c}$, $V^{(\delta_1,0)d}$, and so on, we can easily find the quantities $E_{(\gamma_1,\gamma_2)c.\,(\delta_1,\delta_2)d\cdots}$ defined by

$$E_{(\gamma_1,\gamma_2)c.\,(\delta_1,\delta_2)d\cdots}=\binom{M_2}{\Gamma_2,\ \Delta_2,\ \cdots}E_{\gamma_1 c.\,\delta_1 d\cdots}{}^{[0,\,(M_2-\Gamma_2-\Delta_2)]},$$

which are two parameter extensor components of the order indicated by its Latin indices.

THEOREM 5. *If T^{ab}_{cd} is a tensor of the type* (2, 2, 0), *then the quantities $E^{(\alpha_1,\alpha_2)a.\,(\beta_1,\beta_2)b}_{(\gamma_1,\gamma_2)c.\,(\delta_1,\delta_2)d}$ derived from the tensor T^{ab}_{cd}, are the components of a two parameter extensor of the type* (2, 2, 0, 0, 0).

PROOF. In the generalization of the theorem 3 (Remark 3.2), if we take q equal to 2, then it follows that for arbitrary choice of extensors $U^{(\gamma_1,0)c}$, $V^{(\delta_1,0)d}$, the quantities

$$\begin{bmatrix}A_2,\ B_2\\ M_2\end{bmatrix}\left(U^{(\gamma_1,0)c}V^{(\delta_1,0)d}\ E^{\alpha_1 a.\,\beta_1 b}_{\gamma_1 c.\,\delta_1 d}\right)^{[0,\,(A_2+B_2-M_2)]},$$

constitute the components of a two parameter extensor of the type (2, 2, 0, 0, 0). But

$$\begin{bmatrix}A_2,\ B_2\\ M_2\end{bmatrix}\left(U^{(\gamma_1,0)c}\quad V^{(\delta_1,0)d}\quad E^{\alpha_1 a.\,\beta_1 b}_{\gamma_1 c.\,\delta_1 d}\right)^{[0,\,(A_2+B_2-M_2)]}$$

$$=\sum_{\gamma_2,\,\delta_2=0}\begin{bmatrix}A_2,\ B_2\\ M_2\end{bmatrix}\binom{A_2+B_2-M_2}{\gamma_2,\ \delta_2}U^{(\gamma_1,\gamma_2)c}\quad V^{(\delta_1,\delta_2)d}\ E^{\alpha_1 a.\,\beta_1 b[0,\,(A_2-B_2-M_2-\gamma_2-\delta_2)]}_{\gamma_1 c.\,\delta_1 d}$$

$$=\sum_{\gamma_2,\,\delta_2=0}\begin{Bmatrix}A_2,\ B_2\\ M_2,\gamma_2,\delta_2\end{Bmatrix}U^{(\gamma_1,\gamma_2)c}\quad V^{(\delta_1,\delta_2)d}\quad E^{\alpha_1 a.\,\beta_1 b\,[(0,\,(A_2+B_2-M_2-\gamma_2-\delta_2)]}_{\gamma_1 c.\,\delta_1 d}$$

If we define the quantity $E^{(\alpha_1,\alpha_2)a.\,(\beta_1,\beta_2)b}_{(\gamma_1,\gamma_2)c.\,(\delta_1,\delta_2)d}$ by

$$E^{(\alpha_1,\alpha_2)a.\,(\beta_1,\beta_2)b}_{(\gamma_1,\gamma_2)c.\,(\delta_1,\delta_2)d}=\begin{Bmatrix}A_2,\ B_2\\ M_2,\Gamma_2,\Delta_2\end{Bmatrix}E^{\alpha_1 a.\,\beta_1 b[0,\,(A_2+B_2-M_2-\Gamma_2-\Delta_2)]}_{\gamma_1 c.\,\delta_1 d},$$

$$A_2=\alpha_2,\ \ B_2=\beta_2,\ \ \Gamma_2=\gamma_2,\ \ \Delta_2=\delta_2,$$

the above expression reduces to $U^{(\gamma_1,\gamma_2)c}V^{(\delta_1,\delta_2)d}E^{(\alpha_1,\alpha_2)a.\,(\beta_1,\beta_2)b}_{(\gamma_1,\gamma_2)c.\,(\delta_1,\delta_2)d}$, and the theorem follows from the quotient law. Concerning the range of Greek indices bearing index 2, we, see that the proper range of α_2, β_2, γ_2, δ_2 run from 0 to M_2, but, confining α_2, β_2 to this range, we note in addition that $\begin{bmatrix} A_2, B_2 \\ M_2 \end{bmatrix}$ is zero, if $A_2+B_2<M_2$, so the effective range of $A_2+B_2-M_2$ is from 0 to M_2 inclusively. Further, we take note of $\binom{A_2+B_2-M_2}{\gamma_2, \delta_2}$, that is, whenever $\gamma_2+\delta_2$ exceeds $A_2+B_2-M_2$, the corresponding terms of the sum vanish. Consequently, the maximum range 0 to M_2 may be taken as the actual range for the indices γ_2 and δ_2.

We also notice here that the quantities $E^{\alpha_1 a.\beta_1 b}_{\gamma_1 c.\delta_1 d}$ being appeared as above are defined as

$$E^{\alpha_1 a.\beta_1 b}_{\gamma_1 c.\delta_1 d}=\begin{Bmatrix} A_1, B_1 \\ M_1, \Gamma, \Delta_1 \end{Bmatrix} T^{ab(A_1+B_1-M_1-\Gamma_1-\Delta_1)}_{cd}$$

$$A_1=\alpha_1,\ B_1=\beta_1,\ \Gamma_1=\gamma_1,\ \Delta_1=\delta_1,$$

which are the components of a one parameter extensor of the type (2, 2, 0, 0 ,0) derived from the tensor T^{ab}_{cd} (Craig [3]). In case of the Greek indices α_1, β_1, γ_1 and δ_1, the rule follows similarly for the range as that for the Greek indices bearing index 2 in this theorem.

REMARK 5.1. By the similar process, again constructing the appropriate quantities for the two parameter extensor, and so on, and operating on the set of matrix differentiator $(A_3+B_3-M_3)$, and so on successively, and proceeding as in the proof of this theorem, we can get a k-parameter extensor $E^{(\alpha_1,\alpha_2,\cdots,\alpha_k)a.(\beta_1,\beta_2,\cdots,\beta_k)b}_{(\gamma_1,\gamma_2,\cdots,\gamma_k)c.(\delta_1,\delta_2,\cdots,\delta_k)d}$ of the type (2, 2, 0, 0, 0).

REMARK 5.2. In the proof of the above theorem, if we take the one parameter extensor of unspecified qth order defined by $E^{\alpha_1 a.\beta_1 b.\cdots}_{\gamma_1 c.\delta_1 d}=\begin{Bmatrix} A_1, B_1, \cdots \\ M_1, \Gamma_1, \Delta_1 \end{Bmatrix} T^{ab\cdots[A_1+B_1+\cdots-(q-1)M_1-\Gamma_1-\Delta_1]}_{cd}$ (Craig [3] p. 335), we can easily get a two parameter extensor $E^{(\alpha_1,\alpha_2)a.(\beta_1,\beta_2)b\cdots}_{(\gamma_1,\gamma_2)c.(\delta_1,\delta_2)d}$ of the type (q, 2, 0, 0, 0).

At last, while concluding this paper, the author wishes to acknowledge his grateful thanks to Prof. Dr. Homer V. Craig (University of Texas, Austin) for suggesting the problem, encouragement and many kinds of help in the preparation of this paper.

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