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A DEFINITE INTEGRAL INVOLVING GENERALIZED FOX'S H-FUNCTION WITH APPLICATIONS 1.

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1. Introduction.

In this paper author evaluates a definite integral involving Gauss's hypergeometric function, generalized hypergeometric function and H-function of two variables by means of finite difference operator E and uses it in obtaining a solution of a problem of heat conduction. An expansion formula of very general nature has also been established. Since generalized hypergeometric function may be reduced to simple functions and polynomials and H-function in two arguments is capable of providing generalized Meijer's G-function [1], Fox's H-function, product of two H-functions, generalized Kampe' de Fériet function [17], the results obtained here become master or Key formulae from which a large number of relations can be deduced for functions appearing in applied Mathematics and Mathematical Physics. It is also observed that some recent formulae of Singh [13], [14], [15], Bajpai [2], [3], [4], [5] and Goyal [10] admit themselves of interesting extensions which provide one with the unification of several results scattered throughout the literature.

Mathur [11, p.215] has recently given a generalization of Fox's H-function in two arguments by means of a double Mellin-Barnes countour integral in the form:

$$(1.1)\quad H^{n,\nu_1,\nu_2,m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{array}{c} x \\ \\ y \end{array}\middle|\begin{array}{c} \{(\varepsilon_p,e_p)\} \\ \{(\gamma_t,C_t)\};\ \{(\gamma'_{t'},C'_{t'})\} \\ \{(\delta_s,d_s)\} \\ \{\beta_q,b_q)\};\ \{(\beta'_{q'},b'_{q'})\} \end{array}\right]$$

$$=\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\phi(\xi+\eta)\psi(\xi,\eta)x^{\xi}y^{\eta}d\xi d\eta,$$

where

$$\phi(\xi+\eta)=\frac{\prod_{j=1}^{n}\Gamma(1-\varepsilon_j+e_j\xi+e_j\eta)}{\prod_{j=n+1}^{p}\Gamma(\varepsilon_j-e_j\xi-e_j\eta)\prod_{j=1}^{s}\Gamma(\delta_j+d_j\xi+d_j\eta)}$$

and

$$\phi(\xi,\eta)=\frac{\prod_{j=1}^{m_1}\Gamma(\beta_j-b_j\xi)\prod_{j=1}^{\nu_1}\Gamma(\gamma_j+c_j\xi)\prod_{j=1}^{m_2}\Gamma(\beta'_j-b'_j\eta)\prod_{j=1}^{\nu_2}\Gamma(\gamma'_j+c'_j\eta)}{\prod_{j=m_1+1}^{q}\Gamma(1-\beta_j+b_j\xi)\prod_{j=\nu_1+1}^{t}\Gamma(1-\gamma_j-c_j\xi)\prod_{j=m_2+1}^{q'}\Gamma(1-\beta'_j+b'_j\eta)\prod_{j=\nu_2+1}^{t'}\Gamma(1-\gamma'_j-c'_j\eta)}$$

$\{(A_m, B_m)\}$ stands for a set of m parameters $(A_1, B_1), (A_2, B_2), \cdots, (A_m, B_m)$; $0\le m_1\le q$, $0\le m_2\le q'$, $0\le\nu_1\le t$, $0\le\nu_2\le t'$, $0\le n\le p$.

The sequence of parameters $\{(\beta_{m_1}, b_{m_1})\}$, $\{(\beta'_{m_2}, b'_{m_2})\}$, $\{(\gamma_{\nu_1}, c_{\nu_1})\}$, $\{(\gamma'_{\nu_2}, c'_{\nu_2})\}$ and $\{\varepsilon_n, e_n)\}$ are such that none of the poles of the integrand coincide. The path of integration are indented if necessary, in such a manner that all the poles of $\Gamma(\beta_j-b_j\xi)$ $(j=1,2,\cdots, m_1)$ and $\Gamma(\beta'_k-b'_k\eta)$ $(k=1, 2, \cdots, m_2)$ lie to the right and those of $\Gamma(\gamma_j+c_j\xi)$ $(j=1, 2, \cdots, \nu_1)$, $\Gamma(\gamma'_k+c'_k\eta)$ $(k=1, 2, \cdots, \nu_2)$ and $\Gamma(1-\varepsilon_j+e_j\xi+e_j\eta)$ $(j=1,2, \cdots, n)$ lie to the left of the imaginary axis; the integral converges if $\lambda>0$, $\lambda'>0$, $|\arg x|<\frac{1}{2}\lambda\pi$, $|\arg y|<\frac{1}{2}\lambda'\pi$, where

$$\lambda=\sum_{j=1}^{m_1}b_j+\sum_{j=1}^{\nu_1}c_j+\sum_{j=1}^{h}e_j-\sum_{j=m_1+1}^{q}b_j-\sum_{j=\nu_1+1}^{t}c_j-\sum_{j=n+1}^{p}e_j-\sum_{j=1}^{s}d_j,$$

and

$$\lambda'=\sum_{j=1}^{m_2}b'_j+\sum_{j=1}^{\nu_2}c'_j+\sum_{j=1}^{n}e_j-\sum_{j=m_2+1}^{q'}b'_j-\sum_{j=\nu_2+1}^{t'}c'_j-\sum_{j=n+1}^{p}e_j-\sum_{j=1}^{s}d_j.$$

We shall denote (1.1) symbolically as $H\begin{bmatrix}x\\y\end{bmatrix}$.

The behaviour of $H\begin{bmatrix}x\\y\end{bmatrix}$ for small values of x and y has been discussed by Mathur [11, p.218]:

$H\begin{bmatrix}x\\y\end{bmatrix}=0(|x|^{\beta}|y|^{\beta'})$ as $x\to0$, $y\to0$, where $\beta=\min \mathrm{R}(\beta_h/b_h)$ and $\beta'=\min \mathrm{R}(\beta'_t/b'_t)$ $(h=1,2,\cdots, m_1;\ t=1,2,\cdots,m_2)$ and

$$\sum_{j=1}^{q}b_j-\sum_{j=1}^{t}c_j-\sum_{j=1}^{p}e_j+\sum_{j=1}^{s}d_j\equiv\delta>0,$$

$$\sum_{j=1}^{q'}b'_j-\sum_{j=1}^{t'}c'_j-\sum_{j=1}^{p}e_j+\sum_{j=1}^{s}d_j\equiv\delta'>0.$$

H-function of two variables reduces to Agrawal's G-function in two arguments [1] if e_j $(j=1,2,\cdots,p)$, b_j $(j=1,2,\cdots,q)$, b'_j $(j=1,2,\cdots,q')$, c_j $(j=1,2,\cdots,t)$, c'_j $(j=1,2,\cdots,t')$ and d_j $(j=1,2,\cdots,s)$ are positive integers i.e.

$$(1.2)\qquad H^{n,\nu_1,\nu_2,m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{matrix} x \\ y \end{matrix}\ \middle|\ \begin{matrix} \{(\varepsilon_p,e_p)\} \\ \{(\gamma_t,c_t)\};\ \{(\gamma'_{t'},c'_{t'})\} \\ \{(\delta_s,d_s)\} \\ \{(\beta_q,b_q)\};\ \{(\beta'_{q'},b'_{q'})\} \end{matrix}\right]$$

$$=(2\pi)^D\, PG^{\sum_{j=1}^{n}e_j,\ \sum_{j=1}^{\nu_1}c_j,\ \sum_{j=1}^{\nu_2}c'_j,\ \sum_{j=1}^{m_1}b_j,\ \sum_{j=1}^{m_2}b'_j}_{\sum_{j=1}^{p}e_j,\ \left[\sum_{j=1}^{t}c_j:\sum_{j=1}^{t'}c'_j\right],\ \sum_{j=1}^{s}d_j,\ \left[\sum_{j=1}^{q}b_j:\sum_{j=1}^{q'}b'_j\right]}\left[\begin{matrix} xX_1 \\ yY_1 \end{matrix}\ \middle|\ \begin{matrix} \{\Delta(e_p,\varepsilon_p)\} \\ \{\Delta(c_t,\gamma_t)\};\ \{\Delta(c'_{t'},\gamma'_{t'})\} \\ \{\Delta(d_s,\delta_s)\} \\ \{\Delta(b_q,\beta_q)\};\{\Delta(b'_{q'},\beta'_{q'})\} \end{matrix}\right]$$

where

$$D=\sum_{j=1}^{n}\frac{1-e_j}{2}-\sum_{j=n+1}^{p}\frac{1-e_j}{2}-\sum_{j=1}^{s}\frac{1-d_j}{2}+\sum_{j=1}^{m_1}\frac{1-b_j}{2}-\sum_{j=m_1+1}^{q}\frac{1-b_j}{2}$$
$$+\sum_{j=1}^{\nu_1}\frac{1-c_j}{2}-\sum_{j=\nu_1+1}^{t}\frac{1-c_j}{2}+\sum_{j=1}^{m_2}\frac{1-b'_j}{2}-\sum_{j=m_2+1}^{q'}\frac{1-b'_j}{2}+\sum_{j=1}^{\nu_2}\frac{1-c'_j}{2}-\sum_{j=\nu_2+1}^{t'}\frac{1-c'_j}{2},$$

$$P=\prod_{j=1}^{p}e_j^{\left(\frac{1}{2}-\varepsilon_j\right)}\prod_{j=1}^{s}d_j^{\left(\delta_j-\frac{1}{2}\right)}\prod_{j=1}^{q}b_j^{\left(\beta_j-\frac{1}{2}\right)}\prod_{j=1}^{t}c_j^{\left(\gamma_j-\frac{1}{2}\right)}\prod_{j=1}^{q'}b'_j^{\left(\beta'_j-\frac{1}{2}\right)}\prod_{j=1}^{t'}c'_j^{\left(\gamma'_j-\frac{1}{2}\right)},$$

$$X_1=\frac{\prod_{j=1}^{p}e_j^{e_j}\prod_{j=1}^{t}c_j^{c_j}}{\prod_{j=1}^{s}d_j^{d_j}\prod_{j=1}^{q}b_j^{b_j}},\qquad Y_1=\frac{\prod_{j=1}^{p}e_j^{e_j}\prod_{j=1}^{t'}c'_j^{c'_j}}{\prod_{j=1}^{s}d_j^{d_j}\prod_{j=1}^{q'}b'_j^{b'_j}}$$

and $\Delta(m,a)$ represents a set of m parameters $\frac{a}{m},\ \frac{a+1}{m},\ \cdots,\ \frac{a+m-1}{m}$.

For establishing the integral we shall need the following formulae.

The finite difference operator E [12, p.273 with $h=1$] is

$$(1.3)\qquad E_a f(a)=f(a+1);\ E_a^n f(a)=E_a(E_a^{n-1}f(a)),$$

$$(1.4)\quad \int_0^1 z^{\rho-1}(1-z)^{\beta-1}H^{n,\nu_1,\nu_2,m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{matrix} xz^{m/h} \\ xz^{m/h} \end{matrix}\ \middle|\ \begin{matrix} \{(\varepsilon_p,e_p)\} \\ \{(\gamma_t,c_t)\};\ \{(\gamma'_{t'},c'_{t'})\} \\ \{(\delta_s,d_s)\}, \\ \{(\beta_q,b_q)\};\ \{(\beta'_{q'},b'_{q'})\} \end{matrix}\right]$$

$$=(2\pi)^{(1-h)M}h^N\Gamma(\beta)$$
$$\times H^{nh+1,\nu_1h,\nu_2h,m_1h,m_2h}_{ph+1,[th:t'h],sh+1,[qh:q'h]}\left[\begin{matrix} (xh^{\tau_1})^h \\ (yh^{\tau_2})^h \end{matrix}\ \middle|\ \begin{matrix} (1-\rho,m),\ \{(\Delta(h,\varepsilon_p),e_p)\} \\ \{(\Delta(h,\gamma_t),c_t)\};\{(\Delta(h,\gamma'_{t'}),c'_{t'})\} \\ (\rho+\beta,m),\ \{(\Delta(h,\delta_s),d_s)\} \\ \{(\Delta(h,\beta_q),b_q)\};\{(\Delta(h,\beta'_{q'}),b'_{q'})\} \end{matrix}\right]$$

where m is a positive number and h is a positive integer,

$$M=n+\nu_1+\nu_2+m_1+m_2-\frac{1}{2}(p+s+t+t'+q+q'),$$

$$N=-\sum_{j=1}^{p}\varepsilon_j-\sum_{j=1}^{s}\delta_j+\sum_{j=1}^{q}\beta_j+\sum_{j=1}^{t}\gamma_j+\sum_{j=1}^{q'}\beta'_j+\sum_{j=1}^{t'}\gamma'_j+\frac{1}{2}(p+s-q-t-q'-t')+2,$$

$$\tau_1=\sum_{j=1}^{p}e_j+\sum_{j=1}^{t}c_j-\sum_{j=1}^{s}d_j-\sum_{j=1}^{q}b_j,$$

$$\tau_2=\sum_{j=1}^{p}e_j+\sum_{j=1}^{t'}c'_j-\sum_{j=1}^{s}d_j-\sum_{j=1}^{q'}b'_j$$

and $\{(\Delta(e,a_\gamma),f_\gamma)\}$ represents $\left\{\left(\frac{a_\gamma}{l},f_\gamma\right)\right\},\ \left\{\left(\frac{a_\gamma+1}{l},f_\gamma\right)\right\},\cdots,\left\{\left(\frac{a_\gamma+l-1}{l},f_\gamma\right)\right\}$.

The formula (1.4) holds if $R(\beta)>0, R\left[\rho+\frac{m}{h}(\beta_t/b_t+\beta'_{t'}/b'_{t'})\right]>0$ $(t=1, 2, \cdots, m_1;\ t'=1,2,\cdots,m_2)$, $\delta>0$, $\delta'>0$, $\lambda>0$, $\lambda'>0$, $|\arg x|<\frac{1}{2}\lambda\pi$ and $|\arg y|<\frac{1}{2}\lambda'\pi$.

The formula (1.4) can be obtained by replacing generalized H-function (with $h=1$) on the left hand side by its equivalent countour integral as given in (1.1), changing the order of integration, which is justified due to absolute convergence of the integrals, evaluating the inner integral with the help of [6, p. 9] and applying the multiplication formula for generalized H-function:

$$H^{n,\nu_1,\nu_2,m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{matrix}xz^{m/h}\\ yz^{m/h}\end{matrix}\middle|\begin{matrix}\{(\varepsilon_p,e_p)\}\\ \{(\gamma_t,c_t)\};\{(\gamma'_{t'},c'_{t'})\}\\ \{(\delta_s,d_s)\}\\ \{(\beta_q,b_q)\};\{(\beta'_{q'},b'_{q'})\}\end{matrix}\right]$$

$$=(2\pi)^{(1-h)M}h^N$$

$$\times H^{nh,\nu_1h,\nu_2h,m_1h,m_2h}_{ph,[th:t'h],sh,[qh:q'h]}\left[\begin{matrix}z^m(xh^{\tau_1})^h\\ z^m(yh^{\tau_2})^h\end{matrix}\middle|\begin{matrix}\{(\Delta(h,\varepsilon_p),e_p)\}\\ \{(\Delta(h,\gamma_t),c_t)\};\{(\Delta(h,\gamma'_{t'}),c'_{t'})\}\\ \{(\Delta(h,\delta_s),d_s)\}\\ \{(\Delta(h,\beta_q),b_q)\};\{(\Delta(h,\beta'_{q'}),b'_{q'})\}\end{matrix}\right]$$

2. We shall establish the following integral.

$$(2.1)\quad \int_0^1 z^{\rho-1}(1-z)^{\beta-1}{}_2F_1\left\{\begin{matrix}\alpha_1\nu\\ \beta;\end{matrix}(1-z)\right\}{}_uF_v\left\{\begin{matrix}\alpha_1,\alpha_2,\cdots,\alpha_u\\ \alpha'_1,\alpha'_2,\cdots,\alpha'_v\end{matrix};cz^l\right\}$$

$$\times H^{n,\nu_1,\nu_2,m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{matrix}xz^{m/h}\\ yz^{m/h}\end{matrix}\middle|\begin{matrix}\{(\varepsilon_p,e_p)\}\\ \{(\gamma_t,c_t)\};\{(\gamma'_{t'},c'_{t'})\}\\ \{(\delta_s,d_s)\}\\ \{(\beta_q,b_q)\};\{(\beta'_{q'},b'_{q'})\}\end{matrix}\right]dz$$

$$=(2\pi)^{(1-h)M}h^N\Gamma(\beta)\sum_{\mu=0}^{\infty}\frac{\prod_{j=1}^{u}(\alpha_j)_\mu c^\mu}{\prod_{j=1}^{v}(\alpha'_j)_\mu\mu!}$$

$$\times H^{nh+2,\,\nu_1 h,\,\nu_2 h,\,m_1 h,\,m_2 h}_{ph+2,\,[th:t'h],\,sh+2,\,[qh:q'h]}\left[\begin{matrix}(xh^{\tau_1})^h\\ \\ \\ (yh^{\tau_2})^h\end{matrix}\middle|\begin{matrix}(1-\rho-\mu l,m),(1-\rho-\beta+\alpha+\upsilon-\mu l,m),\\ \{(\Delta(h,\varepsilon_p),e_p)\}\\ \{(\Delta(h,\gamma_t),c_t\};\ \{(\Delta(h,\gamma'_{t'}),c'_{t'})\}\\ (\rho+\beta-\alpha+\mu l,m),(\rho+\beta-\upsilon+\mu l,m),\\ \{(\Delta(h,\delta_s),d_s)\}\\ \{(\Delta(h,\beta_q),b_q)\};\ \{(\Delta(h,\beta'_{q'}),b'_{q'})\}\end{matrix}\right].$$

The conditions of validity for (2.1) are the same as given in (1.4) together with $R(\rho+\beta-\alpha-\upsilon)>0$, $u\leq v$ ($u=v+1$ and $|c|<1$), no one of $\alpha'_1,\alpha'_2,\cdots\alpha'_v$ is zero or a negative integer and l is a positive integer.

Proof of the Integral. On multiplying both sides of (1.4) by

$\dfrac{\prod_{j=1}^{u}\Gamma(\alpha_j+\delta)\Gamma(\alpha)\Gamma(\upsilon)c^\delta}{\prod_{j=1}^{v}\Gamma(\alpha'_j+\delta)\Gamma(\beta)}$, applying the operator $\exp(\overset{l}{E_\rho}E_\delta+E_\alpha E_\beta E_\upsilon)$, expanding both sides, we obtain

$$(2.2)\quad \sum_{\mu=0}^{\infty}\sum_{g=0}^{\infty}\int_0^1\left(\frac{\prod_{j=1}^{u}\Gamma(\alpha_j+\delta+\mu)z^{\rho+\mu l-1}c^{\delta+\mu}}{\prod_{j=1}^{v}\Gamma(\alpha'_j+\delta+\mu)\mu!}\right)\left(\frac{\Gamma(\alpha+g)\Gamma(\upsilon+g)(1-z)^{\beta+g-1}}{\Gamma(\beta+g)\,g!}\right)$$

$$\times H^{n,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{p,\,[t:t'],\,s,\,[q:q']}\left[\begin{matrix}xz^{m/h}\\ \\ yz^{m/h}\\ \end{matrix}\middle|\begin{matrix}\{(\varepsilon_p,e_p)\}\\ \{(\gamma_t,c_t)\};\{(\gamma'_{t'},c'_{t'})\}\\ \{(\delta_s,d_s)\}\\ \{(\beta_q,b_q)\};\{(\beta'_{q'},b'_{q'})\}\end{matrix}\right]dz$$

$$=(2\pi)^{(1-h)M}h^N\sum_{\mu=0}^{\infty}\frac{\prod_{j=1}^{u}\Gamma(\alpha_j+\delta+\mu)c^{\delta+\mu}}{\prod_{j=1}^{v}\Gamma(\alpha'_j+\delta+\mu)\mu!}\sum_{g=0}^{\infty}\frac{\Gamma(\alpha+g)\Gamma(\upsilon+g)}{g!}$$

$$\times H^{nh+1,\,\nu_1 h,\,\nu_2 h,\,m_1 h,\,m_2 h}_{ph+1,\,[th:t'h],\,sh+1,\,[qh:q'h]}\left[\begin{matrix}(xh^{\tau_1})^h\\ \\ (yh^{\tau_2})^h\\ \end{matrix}\middle|\begin{matrix}(1-\rho-\mu l,m),\ \{(\Delta(h,\varepsilon_p),e_p)\}\\ \{(\Delta(h,\gamma_t),c_t)\};\{(\Delta(h,\gamma'_{t'}),c'_{t'})\}\\ (\rho+\beta+\mu l+g,m),\ \{(\Delta(h,\delta_s),d_s)\}\\ \{(\Delta(h,\beta_q),b_q)\};\{(\Delta(h,\beta'_{q'}),b'_{q'})\}\end{matrix}\right].$$

Now changing the order of integration and summation on the left hand side, replacing generalized H-function on the right hand side by its equivalent contour integral as given in (1.1), changing the order of integration and summation, evaluating the inner summation with the help of Gauss's theorem [6, p. 61], interpreting it with the help of (1.1) and replacing $\alpha_j+\delta$ by α_j and $\beta_j+\delta$ by β_j we get (2.1). The change of order of summation and integration involved in the process can easily be justified with the help of [8, p.173, §74 I].

Particular cases:

(i) Putting $c=0$ in (2.1), we obtain

$$(2.3)\quad \int_0^1 z^{\rho-1}(1-z)^{\beta-1}\,{}_2F_1\left\{{\alpha,\upsilon \atop \beta};\ (1-z)\right\} \times H^{n,\nu_1,\nu_2,m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{matrix} xz^{m/h} \\ \\ yz^{m/h} \end{matrix}\left|\begin{matrix} \{(\varepsilon_p,e_p)\} \\ \{(\gamma_t,c_t)\};\{(\gamma'_{t'},c'_{t'})\} \\ \{(\delta_s,d_s)\} \\ \{(\beta_q,b_q)\};\{(\beta'_{q'},b'_{q'})\} \end{matrix}\right.\right] dz$$

$$=(2\pi)^{(1-h)M}h^N\Gamma(\beta) \times H^{nh+2,\nu_1h,\nu_2h,m_1h,m_2h}_{ph+2,[th:t'h],sh+2,[qh:q'h]}\left[\begin{matrix} (xh^{\tau_1})^h \\ \\ (yh^{\tau_2})^h \end{matrix}\left|\begin{matrix} (1-\rho,m),(1-\rho-\beta+\alpha+\upsilon,m),\{(\Delta(h,\varepsilon_p),e_p)\} \\ \{(\Delta(h,\gamma_t),c_t)\};\{(\Delta(h,\gamma'_{t'}),c'_{t'})\} \\ (\rho+\beta-\alpha,m),(\rho+\beta-\nu,m)\{(\Delta(h,\delta_s),d_s)\} \\ \{(\Delta(h,\beta_q),b_q)\};\{(\Delta(h(\beta'_{q'}),b'_{q'})\} \end{matrix}\right.\right]$$

The conditions of validity for (2.3) are the same as specified in (1.4) with $R(\rho+\beta-\alpha-\nu)>0$.

(ii) Replacing α by $-k$ (k is a positive integer), ν by $1+\alpha+\beta+k$, β by $1+\alpha$ in (2.1) and changing ${}_2F_1$ into Jacobi polynomial, we have

$$(2.4)\quad \int_{-1}^1 (1+z)^{\rho-1}(1-z)^{\alpha}P_k^{(\alpha,\beta)}(z)\,{}_uF_v\left\{{\alpha_1,\cdots,\alpha_u \atop \alpha'_1,\cdots,\alpha'_v};c\left(\frac{1+z}{2}\right)^l\right\} \times H^{n,v_1,v_2,m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{matrix} x\left(\frac{1+z}{2}\right)^{m/h} \\ \\ y\left(\frac{1+z}{2}\right)^{m/h} \end{matrix}\left|\begin{matrix} \{(\varepsilon_p,e_p)\} \\ \{(\gamma_t,c_t)\};\{\gamma'_{t'},c'_{t'})\} \\ \{(\delta_s,d_s)\} \\ \{(\beta_q,b_q)\};\{(\beta'_{q'},b'_{q'})\} \end{matrix}\right.\right] dz$$

$$=(2\pi)^{(1-h)M}h^N2^{\rho+\alpha}\frac{\Gamma(1+\alpha+k)}{k!}\sum_{\mu=0}^{\infty}\frac{\prod_{j=1}^{u}(\alpha_j)_\mu c^\mu}{\prod_{j=1}^{v}(\alpha'_j)_\mu \mu!}$$

$$\times H^{nh+2,\nu_1h,\nu_2h,m_1h,m_2h}_{ph+2,[th:t'h],sh+2,[qh:q'h]}\left[\begin{matrix} (xh^{\tau_1})^h \\ \\ \\ (yh^{\tau_2})^h \end{matrix}\left|\begin{matrix} (1-\rho-\mu l,m),(1-\rho+\beta-\mu l,m),\{(\Delta(h,\varepsilon_p),e_p)\} \\ \{(\Delta(h,\gamma_t),c_t)\};\{(\Delta(h,\gamma'_{t'}),c'_{t'})\} \\ (1+\rho+\alpha+k+\mu l,m),(\rho-\beta-k+\mu l,m), \\ \{(\Delta,h,\delta_s),d_s)\} \\ \{(\Delta(h,\beta_q),b_q)\};\{(\Delta(h,\beta'_{q'}),b'_{q'})\} \end{matrix}\right.\right]$$

The equation (2.4) is valid under the same conditions as given in (1.4) along with $u \leq v$ ($u=v+1$ and $|c|<1$), no one of $\alpha'_1, \alpha'_2, \cdots, \alpha'_v$ is zero or a negative integer and l is a positive integer.

(iii) Next take $p=n$, $q'=m_2=1$, $\beta'_1=0, b'_1=1$, $\nu_2=t'=0$ in (2.1) and make $y \to 0$, then replace $n+\nu_1$ by m_2, $t+n$ by t, $q+s$ by q together with the appropriate changes in the parameters, we get

$$(2.5) \int_0^1 z^{\rho-1}(1-z)^{\beta-1} {}_2F_1\left\{\begin{matrix}\alpha, \nu\\ \beta\end{matrix}; (1-z)\right\} {}_uF_v\left\{\begin{matrix}\alpha_1, \cdots, \alpha_u\\ \alpha'_1, \cdots, \alpha'_v\end{matrix}; cz^l\right\} H_{t,q}^{m_1, m_2}\left[xz^{m/h} \middle| \begin{matrix}\{(\varepsilon_t, e_t)\}\\ \{(\beta_q, b_q)\}\end{matrix}\right] dz$$

$$= (2\pi)^{(1-h)M} h^N \Gamma(\beta) \sum_{\mu=0}^{\infty} \frac{\prod_{j=1}^{u} (\alpha_j)_\mu c^\mu}{\prod_{j=1}^{v} (\alpha'_j)_\mu \mu!}$$

$$\times H_{th+2, qh+2}^{m_1 h, m_2 h+2}\left[(xh^\tau)^h \middle| \begin{matrix}(1-\rho-\mu l, m), (1-\rho-\beta+\alpha+\nu-\mu l, m), \{(\Delta(h, \varepsilon_t), e_t)\}\\ \{(\Delta(h, \beta_q), b_q)\}, (1+\alpha-\rho-\beta-\mu l, m), (1+\nu-\rho-\beta-\mu l, m)\end{matrix}\right],$$

provided m is a positive number and h is a positive integer,

$R(\beta)>0$, $R\left(\rho+\frac{m}{h}\frac{\beta_t}{b_t}\right)>0$ $(t=1, 2, \cdots, m_1)$, $R(\rho+\beta-\alpha-\nu)>0$,

$T \equiv \sum_{j=1}^{t} e_j - \sum_{j=1}^{q} b_j \leq 0$, $\sum_{j=1}^{m_2} e_j - \sum_{j=m_2+1}^{t} e_j + \sum_{j=1}^{m_1} b_j - \sum_{j=m_1+1}^{q} b_j \equiv M_1 > 0$, $|\arg x| < \frac{1}{2}\pi M_1$; $u \leq v$ ($u=v+1$ and $|c|<1$), no one of $\alpha'_1, \alpha'_2, \cdots, \alpha'_v$ is zero or a negative integer and l is a positive integer.

Further on reducing ${}_2F_1$ into Jacobi polynomial and H-function into Meijer's G-function we get a result of Singh [13, p.156, 2.1] which, in turn, on certain manipulations yields a recent result of Bajpai [2, p.113, (2.1)].

Again putting $c=0$ in (2.5) and reducing ${}_2F_1$ into Jacobi Polynomial we obtain a result given by Bajpai [3, (3.1)]

(iv) On setting e_j $(j=1, 2, \cdots, p) = b_j(j=1, 2, \cdots, q) = b'_j(j=1, 2 \cdots, q') = c_j(j=1, 2, \cdots t) = c'_j(j=1, 2, \cdots, t') = d_j(j=1, 2, \cdots, s) = 1$ in (2.1) and reducing H-function of two variables to G-function in two arguments with the help of (1.2), we get the recent result due to Singh [14, (2.1)].

Further with $c=0$, we obtain a result given by Singhal [16, p.981].

3. Expansion Formula.

We shall establish the following expansion formula

$$(3.1)\quad (1+z)^{\rho-1}{}_uF_v\left\{\begin{matrix}\alpha_1,\cdots,\alpha_u\\ \alpha'_1,\cdots,\alpha'_v\end{matrix};c\left(\frac{1+z}{2}\right)^l\right\}$$

$$\times H^{n,\nu_1,\nu_2,m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{matrix}x\left(\frac{1+z}{2}\right)^{m/h}\\ y\left(\frac{1+z}{2}\right)^{m/h}\end{matrix}\left|\begin{matrix}\{(\varepsilon_p,e_p)\}\\ \{(\gamma_t,c_t)\};\{(\gamma'_{t'},c'_{t'})\}\\ \{(\delta_s,d_s)\}\\ \{(\beta_q,b_q)\};\{(\beta'_{q'},b'_{q'})\}\end{matrix}\right.\right]$$

$$=(2\pi)^{(1-h)M}h^N2^{\rho-1}\sum_{\mu,\gamma=0}^{\infty}\frac{(1+\alpha+\beta+2\gamma)\Gamma(1+\alpha+\beta+\gamma)}{\Gamma(1+\beta+\gamma)}\frac{\prod_{j=1}^{u}(\alpha_j)_\mu c^\mu}{\prod_{j=1}^{v}(\alpha'_j)_\mu\mu!}$$

$$\times H^{nh+2,\nu_1h,\nu_2h,m_1h,m_2h}_{ph+2,[th:t'h],sh+2,[qh:q'h]}\left[\begin{matrix}(xh^{\tau_1})^h\\ \\ \\ (yh^{\tau_2})^h\end{matrix}\left|\begin{matrix}(1-\rho-\beta-\mu l,m),(1-\rho-\mu l,m),\\ \{(\Delta(h,\varepsilon_p),e_p)\}\\ \{(\Delta(h,\gamma_t),c_t)\};\{(\Delta(h,\gamma'_{t'}),c'_{t'})\}\\ (1+\rho+\alpha+\beta+\gamma+\mu l,m),\\ (\rho-\gamma+\mu l,m),\{(\Delta(h,\delta_s),d_s)\}\\ \{(\Delta(h,\beta_q),b_q)\};\{(\Delta(h,\beta'_{q'}),b'_{q'})\}\end{matrix}\right.\right]P_r^{(\alpha,\beta)}(z).$$

The conditions of validity for (3.1) are the same as given in (2.4) with $\rho=\rho+\beta$ along with $\mathrm{Re}(\alpha)>-1$ and $\mathrm{Re}(\beta)>-1$.

PROOF. Let

$$(3.2)\quad f(z)=(1+z)^{\rho-1}{}_uF_v\left\{\begin{matrix}\alpha_1,\cdots,\alpha_u\\ \alpha'_1,\cdots,\alpha'_v\end{matrix};c\left(\frac{1+z}{2}\right)^l\right\}$$

$$\times H^{n,\nu_1,\nu_2,m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{matrix}x\left(\frac{1+z}{2}\right)^{m/h}\\ y\left(\frac{1+z}{2}\right)^{m/h}\end{matrix}\left|\begin{matrix}\{(\varepsilon_p,e_p)\}\\ \{(\gamma_t,c_t)\};\{(\gamma'_{t'},c'_{t'})\}\\ \{(\delta_s,d_s)\}\\ \{(\beta_q,b_q)\};\{(\beta'_{q'},b'_{q'})\}\end{matrix}\right.\right]$$

$$=\sum_{r=0}^{\infty}A_rP_r^{(\alpha,\beta)}(z),\ -1<z<1.$$

Equation (3.2) is valid since $f(z)$ is continuous and of bounded variation in the open interval $(-1,1)$ when $\rho\geq 1$. Multiply both sides of (3.2) by $(1-z)^\alpha(1+z)^\beta$ $p_k^{(\alpha,\beta)}(z)$ and integrate with respect to z from -1 to 1, use (2.4) and the orthogonal property of Jacobi polynomial [7, p.285, (5) and (9)], we obtain

$$(3.3)\quad A_k=(2\pi)^{(1-h)M}h^N2^{\rho-1}\frac{(1+\alpha+\beta+2k)\Gamma(1+\alpha+\beta+k)}{\Gamma(1+\beta+k)}\sum_{\mu=0}^{\infty}\frac{\prod_{j=1}^{u}(\alpha_j)_\mu c^\mu}{\prod_{j=1}^{v}(\alpha'_j)_\mu\mu!}$$

$$\times H^{nh+2,\nu_1 h,\nu_2 h,m_1 h,m_2 h}_{ph+2,[th:t'h],sh+2,[qh:q'h]}\left[\begin{array}{c|l}(xh^{\tau_1})^h & (1-\rho-\beta-\mu l,m),(1-\rho-\mu l,m),\{(\Delta(h,\varepsilon_p),e_p)\} \\ & \{(\Delta(h,\gamma_t),c_t)\};\{(\Delta(h,\gamma'_{t'}),c'_{t'})\} \\ & (1+\rho+\alpha+\beta+k+\mu l,m),(\rho-k+\mu l,m), \\ (yh^{\tau_2})^h & \{(\Delta(h,\delta_s),d_s)\} \\ & \{(\Delta(h,\beta_q),b_q)\};\{(\Delta(h,\beta'_{q'}),b'_{q'})\}\end{array}\right]$$

On substituting the value of A_k from (3.3) in (3.2) we get (3.1).

On specializing the parameters we get the following results.

(i) If we take $p=s=c=0$ in (3.1), generalized H-function breaks up into the product of two H-functions and thus we have

$$(3.4)\quad (1+z)^{\rho-1}H^{m_1,\nu_1}_{t,q}\left[x\left(\frac{1+z}{2}\right)^{m/h}\left|\begin{array}{l}\{(1-\gamma_t,c_t)\}\\ \{\beta_q,b_q)\}\end{array}\right.\right]H^{m_2,\nu_2}_{t',q'}\left[y\left(\frac{1+z}{2}\right)^{m/h}\left|\begin{array}{l}\{(1-\gamma'_{t'},c'_{t'})\}\\ \{(\beta'_{q'},b'_{q'})\}\end{array}\right.\right]$$

$$=(2\pi)^{(1-h)M}h^{N}2^{\rho-1}\sum_{\gamma=0}^{\infty}\frac{(1+\alpha+\beta+2\gamma)\Gamma(1+\alpha+\beta+\gamma)}{\Gamma(1+\beta+\gamma)}$$

$$\times H^{2,\nu_1 h,\nu_2 h,m_1 h,m_2 h}_{2,[th:t'h],2,[qh:q'h]}\left[\begin{array}{c|l}(xh^{\tau_1})^h & (1-\rho-\beta,m),(1-\rho,m) \\ & \{(\Delta(h,\gamma_t),c_t)\};\{(\Delta(h,\gamma'_{t'}),c'_{t'})\} \\ (yh^{\tau_2})^h & (1+\rho+\alpha+\beta+\gamma,m),(\rho-\gamma,m) \\ & \{(\Delta(h,\beta_q),b_q)\};\{(\Delta(h,\beta'_{q'}),b'_{q'})\}\end{array}\right]$$

where conditions of validity being the same as given in (3.1) with $p=s=c=0$.

(ii) If we put $e_j(j=1,2,\cdots,p)=b_j(j=1,2,\cdots,q)=b'_j(j=1,2,\cdots,q')=c_j(j=1,2,\cdots,t)=c'_j(j=1,2,\cdots,t')=d_j(j=1,2,\cdots,s)=h=1$ and $c=0$ in (3.1), then in the light of the equation (1.2) we get a recent result of Goyal [9, p.227, 4.1].

(iii) Next take $p=n$, $q'=m_2=1$, $\beta'_1=0$, $b'_1=1$, $\nu_2=t'=0$ in (3.1) and make $y\to 0$, then replace $n+\nu_1$ by m_2, $t+n$ by t, $q+s$ by q together with the appropriate changes in the parameters we obtain a known result due to Singh [15, (2.1)].

Further on taking $c=0$, we get a result given by Bajpai [4, p.23, (1.2)].

4. Heat conduction.

We shall consider the problem of determining a function $\phi(z,w)$ which represents the temperature in a non-homogeneous bar with ends at $z=-1$ to $z=1$ in which the thermal conductivity is proportional to $(1-z^2)$ and if the lateral surface of the bar is insulated, the heat equation has the form (9, p.197, (8)]

$$(4.1)\qquad \frac{\partial\phi}{\partial w}=b\frac{\partial}{\partial z}\left[(1-z^2)\frac{\partial\phi}{\partial z}\right],$$

where b is constant provided the thermal coefficient is constant [9, p.17, sec. 9]. The ends $z\pm 1$ are also insulated because the conductivity vanishes there and

$$(4.2)\qquad \phi|_{w=0}=f(z)$$

The solution of (4.1) to be obtained is

$$(4.3)\quad \phi(z,w)=(2\pi)^{(1-h)M}h^{N}2^{\rho-1}\sum_{\nu,\mu=0}^{\infty}(2\nu+1)P_{\nu}(z)e^{-b\nu(\nu+1)w}\frac{\prod_{j=1}^{u}(\alpha_j)_{\mu}c^{\mu}}{\prod_{j=1}^{v}(\alpha'_j)_{\mu}\mu!}$$

$$\times H^{nh+2,\nu_1h,\nu_2h,m_1h,m_2h}_{ph+2,[th:t'h],sh+2,[qh:q'h]}\left[\begin{matrix}(xh^{\tau_1})^h\\ \\ (yh^{\tau_2})^h\end{matrix}\left|\begin{matrix}(1-\rho-\mu l,m),(1-\rho-\mu l,m),\{(\Delta(h,\varepsilon_p),e_p)\}\\ \{(\Delta(h,\gamma_t),c_t)\};\{(\Delta(h,\gamma'_{t'}),c'_{t'})\}\\ \{1+\rho+\nu+\mu l,m),(\rho-\nu+\mu l,m),\{(\Delta(h,\delta_s),d_s)\}\\ \{(\Delta(h,\beta_q),b_q)\};\{(\Delta(h,\beta'_{q'}),b'_{q'})\}\end{matrix}\right.\right],$$

the conditions of validity being the same as given with (2.4) with $\alpha=\beta=0$.

Proof of the solution (4.3). The solution of (4.1) as given in Churchil [9, p. 198, (8)] is

$$(4.4)\qquad \phi(z,w)=\sum_{\nu=0}^{\infty}M_{\nu}P_{\nu}(z)e^{-b\nu(\nu+1)w}$$

Because of the condition (4.2) the coefficients of M_{ν} must be chosen to satisfy the relation

$$(4.5)\qquad f(z)=\sum_{\nu=0}^{\infty}M_{\nu}P_{\nu}(z).$$

If we take

$$(4.6)\quad f(z)=(1+z)^{\rho-1}{}_uF_v\left\{\begin{matrix}\alpha_1,\cdots,\alpha_u\\ \alpha'_1,\cdots,\alpha'_v\end{matrix};c\left(\frac{1+z}{2}\right)^l\right\}$$

$$\times H^{n,\nu_1,\nu_2,m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{matrix}x\left(\frac{1+z}{2}\right)^{m/h}\\ \\ y\left(\frac{1+z}{2}\right)^{m/h}\end{matrix}\left|\begin{matrix}\{(\varepsilon_p,e_p)\}\\ \{(\gamma_t,c_t)\};\{(\gamma'_{t'},c'_{t'})\}\\ \{(\delta_s,d_s)\}\\ \{(\beta_q,b_q)\};\{(\beta'_{q'},b'_{q'})\}\end{matrix}\right.\right],$$

we get

$$(4.7)\quad \sum_{\nu=0}^{\infty}M_{\nu}P_{\nu}(z)=(1+z)^{\rho-1}{}_uF_v\left\{\begin{matrix}\alpha_1,\cdots,\alpha_u\\ \alpha'_1,\cdots,\alpha'_v\end{matrix};c\left(\frac{1+z}{2}\right)^l\right\}$$

$$\times H^{n,\nu_1,\nu_2,m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{matrix}x\left(\frac{1+z}{2}\right)^{m/h}\\ \\ y\left(\frac{1+z}{2}\right)^{m/h}\end{matrix}\left|\begin{matrix}\{(\varepsilon_p,e_p)\}\\ \{(\gamma_t,c_t)\};\{(\gamma'_{t'},c'_{t'})\}\\ \{(\delta_s,d_s)\}\\ \{(\beta_q,b_q)\};\{(\beta'_{q'},b'_{q'})\}\end{matrix}\right.\right].$$

Multiply both sides of (4.7) by $P_k(z)$ and integrate with respect to z between the limit $s-1$ and 1, we have

$$(4.8)\quad \sum_{\nu=0}^{\infty}M_{\nu}\int_{-1}^{1}P_{\nu}(z)P_k(z)dz=\int_{-1}^{1}(1+z)^{\rho-1}P_k(z){}_uF_v\left\{\begin{matrix}\alpha_1,\cdots,\alpha_u\\ \alpha'_1,\cdots,\alpha'_v\end{matrix};c\left(\frac{1+z}{2}\right)^l\right\}$$

$$\times H^{n,\nu_1,\nu_2,m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{matrix} x\left(\frac{1+z}{2}\right)^{m/h} \\ y\left(\frac{1+z}{2}\right)^{m/h} \end{matrix}\middle| \begin{matrix} \{(\varepsilon_p, e_p)\} \\ \{(\gamma_t, c_t)\};\{(\gamma'_{t'}, c'_{t'})\} \\ \{(\delta_s, d_s)\} \\ \{(\beta_q, b_q)\};\{(\beta'_{q'}, b'_{q'})\} \end{matrix}\right] dz$$

Now use (2.4) with $\alpha=\beta=0$ and orthogonal property of Legendre's polynomial [7, p.277, (13) and (14)], we get

$$(4.9)\quad M_k=\frac{2k+1}{2}(2\pi)^{(1-h)M}h^N 2^\rho \sum_{\mu=0}^{\infty} \frac{\prod_{j=1}^{\mu}(\alpha_j)_\mu c^\mu}{\prod_{j=1}^{v}(\alpha'_j)_\mu \mu!}$$

$$\times H^{nh+2,\nu_1 h,\nu_2 h,m_1 h,m_2 h}_{ph+2,[th:t'h],sh+2,[qh:q'h]}\left[\begin{matrix} (xh^{\tau_1})^h \\ (yh^{\tau_2})^h \end{matrix}\middle| \begin{matrix} (1-\rho-\mu l, m),(1-\rho-\mu l, m),\{(\Delta(h,\varepsilon_p),e_p)\} \\ \{(\Delta(h,\gamma_t),c_t)\};\{(\Delta(h,\gamma'_{t'}),c'_{t'})\} \\ (1+\rho+k+\mu l, m),(\rho-k+\mu l, m),\{(\Delta(h,\delta_s),d_s)\} \\ \{(\Delta(h,\beta_q),b_q)\};\{(\Delta(h,\beta'_{q'}),b'_{q'})\} \end{matrix}\right],$$

Now with the help of (4.4) and (4.9), we arrive at (4.3).

Proceeding on parallel lines as in the case of equation (2.5) with $c=0$ and reducing H-function into Meijer's G-function, the equation (4.3) takes the form of a result obtained by Bajpai [5].

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