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A SUMMATION FORMULA ON KAMPÉ DE FÉRIET FUNCTION

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During the course of investigation, an attempt has been made by several workers to obtain the exact solution of many problems in quantum mechanics in terms of Appell's functions; in an extension of this work, the author establishes here a summation formula for Kampé de Fériet's function, which may prove to be useful. The formula is as follows:

$$\sum_{m=0}^{n}\frac{\Gamma\left(k+m+\frac{1}{2}\right)(2yz)^m}{m!}F\left[\begin{array}{c|c|c} 1 & -k-m & \\ 2 & \Delta(2,-m);\ \Delta(2,-m) & y^{-2} \\ 2 & \Delta(2,-2k-2m) & z^{-2} \\ 0 & -;- & \end{array}\right]$$

$$=\frac{(2yz)^n\Gamma\left(k+n+\frac{1}{2}\right)}{n!(y-z)}\left\{yF\left[\begin{array}{c|c|c} 1 & -k-n & \\ 2 & \Delta(2,-n-1);\Delta(2,-n) & y^{-2} \\ 2 & \Delta(2,-2k-2n) & z^{-2} \\ 0 & -;- & \end{array}\right]-\rightleftharpoons\right\}$$

where "$\rightleftharpoons$" shows the presence of a similar term with y and z interchanged.

1. Introduction.

Kampé de Fériet J. [(1), p.150] has introduced a generalized hypergeometric function of two variables in the form

$$(1.1)\qquad F\left[\begin{array}{c|c|c} m & a_m & \\ l & b_l;b_l' & x \\ n & c_n & y \\ p & d_p;d_p' & \end{array}\right]=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\prod_{j=1}^{m}(a_j)_{r+s}\prod_{j=1}^{l}\{(b_j)_r(b_j')_s\}x^r y^s}{r!s!\prod_{j=1}^{n}(c_j)_{r+s}\prod_{j=1}^{p}\{(d_j)_r(d_j')_s\}},$$

where a_m stands for $a_1,a_2,\cdots,a_m$; $\prod_{j=1}^{m}(a_j)_r$ represents the product $(a_1)_r(a_2)_r\cdots(a_m)_r$ and for absolute convergence of the series $|x|<1$, $|y|<1$, $m+l\leq n+p+1$.

The aim of this note is to establish a summation formula for Kampé de Fériet's function which may be utilized in the exact solution of a large number of problems of Mathematics, both pure and applied, and in mathematical physics frequently expressed in terms of Kampé de Fériet's functions. In a recent paper [(3), p. 9, (4.5)] the author has obtained the formal solution of a problem of heat conduction equation in terms of Kampé de Fériet's functions.

2. A Summation Formula.

We start from the known result [(3), p. 5, (2.1) for $n=0$]:

$$(2.1)\quad \int_{-\infty}^{\infty} e^{-x^2}\, x^{2k}\Big[\Big\{x^{l}\,{}_{p+2}F_q\Big(\Delta(2,-l), A_p; B_q; \mu x^{-2}\Big)\Big\}\Big\{x^{m}\,{}_{p+2}F_q\Big(\Delta(2,-m), a_p; b_q; \lambda x^{-2}\Big)\Big\}\Big]dx$$

$$=\Gamma\Big(k+\frac{1}{2}m+\frac{1}{2}l+\frac{1}{2}\Big)\,F\left[\begin{array}{c|c|c} 1 & -k-\frac{1}{2}m-\frac{1}{2}l & -\mu \\ p+2 & \Delta(2,-l),\ A_p;\ \Delta(2,-m), a_p & \\ 2 & \Delta(2,-2k-m-l) & -\lambda \\ q & B_q;\ b_q & \end{array}\right]$$

where l, m are positive integers and $\Delta(m, n)=\frac{n}{m}, \frac{n+1}{m}, \ldots, \frac{n+m-1}{m}$.

Further setting $p=q=0$, $\mu=-y^{-2}$, $\lambda=-z^{-2}$, $l=m$ etc, in (2.1), we get

$$(2.2)\quad \int_{-\infty}^{\infty} e^{-x^2} x^{2k} H_m(yx)H_m(zx)dx=\Gamma\Big(k+m+\frac{1}{2}\Big)(4yz)^m$$

$$F\left[\begin{array}{c|c|c} 1 & -k-m & y^{-2} \\ 2 & \Delta(2,-m);\ \Delta(2,-m) & \\ 2 & \Delta(2,-2k-2m) & z^{-2} \\ 0 & -;- & \end{array}\right]$$

where $H_n(x)=(2x)^n\,{}_2F_0\left[\Delta(2,-n); -; -x^{2}\right]$ is the Hermite polynomial.

Therefore

$$\sum_{m=0}^{n}\frac{\Gamma\Big(k+m+\frac{1}{2}\Big)(2yz)^m}{m!}\,F\left[\begin{array}{c|c|c} 1 & -k-m & y^{-2} \\ 2 & \Delta(2,-m);\ \Delta(2,-m) & \\ 2 & \Delta(2,-2k-2m) & z^{-2} \\ 0 & -;- & \end{array}\right]$$

$$=\sum_{m=0}^{n}\frac{1}{2^m m!}\int_{-\infty}^{\infty} e^{-x^2} x^{2k} H_m(yx)H_m(zx)dx.$$

The change in order of summation and integration is easily justified and we have the R.H.S. as

$$=\int_{-\infty}^{\infty} e^{-x^2}\, x^{2k}\Big\{\sum_{m=0}^{n}\frac{H_m(yx)H_m(zx)}{2^m m!}\Big\}dx.$$

Using, Christoffel-Darboux formula [(2), p. 193, (11)]:

$$\sum_{m=0}^{n}\frac{H_m(x)H_m(y)}{2^m m!}=\frac{H_{n+1}(x)H_n(y)-H_n(x)H_{n+1}(y)}{2^{n+1}\,n!\,(x-y)},$$

the R.H.S. becomes

$$=\frac{(y-z)^{-1}}{2^{n+1}n!}\int_{-\infty}^{\infty}e^{-x^2}x^{2k-1}[H_{n+1}(yx)H_n(zx)-H_n(yx)H_{n+1}(zx)]\,dx.$$

Now seperating the R.H.S. as the difference of two integrals then in view of (2.1), we obtain

$$(2.3)\qquad \sum_{m=0}^{n}\frac{\Gamma\left(k+m+\frac{1}{2}\right)(2yz)^m}{m!}F\left[\begin{array}{c|c|c} 1 & -k-m & y^{-2} \\ 2 & \Delta(2,-m);\ \Delta(2,-m) & \\ 2 & \Delta(2,-2k-2m) & z^{-2} \\ 0 & -;- & \end{array}\right]$$

$$=\frac{(2yz)^n\Gamma\left(k+n+\frac{1}{2}\right)}{n!(y-z)}\left\{yF\left[\begin{array}{c|c|c} 1 & -k-n & y^{-2} \\ 2 & \Delta(2,-n-1);\ \Delta(2,-n) & \\ 2 & \Delta(2,-2k-2n) & z^{-2} \\ 0 & -;- & \end{array}\right]-\rightleftharpoons\right\}$$

where "$\rightleftharpoons$" is used to indicate the presence of a similar term with y and z interchanged. This is the summation formula.

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[3] Shah Manilal, *On applications of Hermite polynomials*, Journal of Pure and Applied Sciences, Middle East Technical University, Ankara-Turkey, (In press).