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A PROPERTY OF COFUNCTORS $S_F(X, A)$

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Abstract.

A k-dimensional vector bundle is a bundle $\xi=(E, P, B, F^k)$ with fibre F^k satisfying the *local triviality*, where F is the field of real numbers $\boldsymbol{R}$ or complex numbers $\boldsymbol{C}$ ([1], [2] and [3]). Let $\mathrm{Vect}_k(X)$ be the set consisting of all isomorphism classes of k-dimensional vector bundles over the topological space X. Then $\mathrm{Vect}_F(X)=\{\mathrm{Vect}_k(X)\}_{k=0,1,\cdots}$ is a semigroup with Whitney sum (§1).

For a pair (X, A) of topological spaces, a *difference isomorphism* over (X, A) is a vector bundle morphism ([2], [3]) α: $\xi_0 \to \xi_1$ such that the restriction α: $\xi_0|A \longrightarrow \xi_1|A$ is an isomorphism. Let $S_k(X, A)$ be the set of all difference isomorphism classes over (X, A) of k-dimensional vector bundles over X with fibre F^k. Then $S_F(X, A)=\{S_k(X, A)\}_{k=0,1,\cdots,}$ is a semigroup with Whitney sum (§2).

In this paper, we shall prove a relation between $\mathrm{Vect}_F(X)$ and $S_F(X, A)$ under some conditions (Theorem 2, which is the main theorem of this paper). We shall use the following theorem in the paper.

THEOREM 1. *Let $\xi=(E, P, B)$ be a locally trivial bundle with fibre F, where (B, A) is a relative CW-complex. Then all cross sections S of $\xi|A$ prolong to a cross section S^* of ξ under either of the following hypothesis:*

(H1) *The space F is $(m-1)$-connected for each $m \leq \dim B$.*

(H2) *There is a relative CW-complex (Y, X) such that $B=Y\times I$ and $A=(X\times I) \cap (Y\times 0)$, where $I=[0,1]$.* (For proof see p.21 [2]).

1. Cofunctors Vect_k.

Let $\xi=(E(\xi), P_\xi, B)$ and $\eta=(E(\eta), P_\eta, B)$ be two vector bundles over B. We define $\xi\oplus\eta$ by

$$E(\xi\oplus\eta)=\bigcup_{b\in B}\xi_b\oplus\eta_b$$

then, with a suitable topology on $E(\xi\oplus\eta)$, $\xi\oplus\eta=(E(\xi\oplus\eta), P_\xi\oplus P_\eta, B)$

is a vector bundle over B, where $\xi_b=P_\xi^{-1}(b)$ and $\eta_b=P_\eta^{-1}(b)$. ([1], [2]). We call $\xi\oplus\eta$ the Whitney sum of ξ and η.

For a continuous map f: $A\longrightarrow B$, and a vector bundle ξ over B, we define the *fibre products bundle* $f^*(\xi)$ induced by f as follows,

$$E(f^*(\xi))=\{(x,y)\in A\times E(\xi)\,|\,f(x)=P(y)\},$$

$$P_{f^*(\xi)}:\ E(f^*(\xi))\longrightarrow A \text{ is defined by } P_{f^*(\xi)}((x,y))=x.$$

Then, $f^*(\xi)=(E(f^*(\xi)),\ P_{f^*(\xi)},\ A)$ is a vector bundle over A ([1], [3]).

Let **Pa** be the category of paracompact spaces and homotopy classes of maps, and let **Ens** be the category of sets and functions. We take $[f]:A\to B$ in **Pa**. For $f,g\in[f]$ and a vector bundle ξ over B, we can prove that $f^*(\xi)\cong g^*(\xi)$ ([2]). Now, we define a function Vect_k: **Pa**$\to$**Ens** by $\mathrm{Vect}_k([f])$: $\mathrm{Vect}_k(B)\longrightarrow\mathrm{Vect}_k(A)$ such that $[\mathrm{Vect}_k([f])](\{\xi\})=\{f^*(\xi)\}$ in **Ens** for $[f]$: $A\longrightarrow B$ in **Pa**, where $\{\xi\}\in\mathrm{Vect}_k(B)$ and $\{f^*(\xi)\}\in\mathrm{Vect}_k(A)$ are the isomorphism classes containing ξ. For f, $g\in[f]:A\longrightarrow B$, since $f^*(\xi)\cong g^*(\xi)$ $\mathrm{Vect}_k([f])$ is well defined.

PROPOSITION 1. *The family of functions* Vect_k: **Pa**$\longrightarrow$**Ens** *is a cofunctor.*

PROOF. For 1_B: $B\longrightarrow B$, $B\in$ **Pa**, and a vector bundle ξ over B, $1_B^*(\xi)\cong\xi$ and therefore $\mathrm{Vect}_k([1_B])$ is the identity. If $g\circ f$: $B_2\xrightarrow{f}B_1\xrightarrow{g}B$ is continuous, where B_2, B_1 and $B\in$ **Pa**, for a vector bundle ξ over B, $(g\circ f)^*(\xi)=f^*\circ g^*(\xi)$ ([2]). Thus $\mathrm{Vect}_k([g\circ f])=\mathrm{Vect}_k([f])\circ\mathrm{Vect}_k([g])$. Therefore Vect_k is a cofunctor.

The Stiefel variety of orthogonal k-frames in $\boldsymbol{R}^n$, written $V_k(\boldsymbol{R}^n)$, is the subspace of $(v_1,\cdots,v_k)\in(S^{n-1})^k$ such that $v_i\perp v_j$ for $i\neq j$. The *Grassmann variety* of k-dimensional subspaces of $\boldsymbol{R}^n$, written $G_k(\boldsymbol{R}^n)$, is the set of k-dimensional subspaces of $\boldsymbol{R}^n$ with the quotient topology defined by the function $(v_1,\cdots,v_k)\longrightarrow\langle v_1,\cdots,v_k\rangle$ of $V_k(\boldsymbol{R}^n)$ onto $G_k(\boldsymbol{R}^n)$, where $\langle v_1,\cdots,v_k\rangle$ is the k-dimensional subspace of $\boldsymbol{R}^n$ with basis $v_1,\cdots,v_k$ ([2]). We have the canonical k-dimensional vector bundle γ_k^n over $G_k(\boldsymbol{R}^n)$, which is defined by $E(\gamma_k^n)=\{(V,x)\in G_k(\boldsymbol{R}^n)\times\boldsymbol{R}^n\,|\,x\in V\}$, $P_{\gamma_k^n}$: $E(\gamma_k^n)\to G_k(\boldsymbol{R}^n)$ is the projection on the first argument, that is $\gamma_k^n=(E(\gamma_k^n),\ P_{\gamma_k^n},G_k(\boldsymbol{R}^n))$. γ_k^n is the vector bundle with fibre $\boldsymbol{R}^k$, i.e γ_k^n is the

k-dimensional vector bundle over $G_k(R^n)$.

Suppose a space B is in **Pa**. Every k-dimensional vector bundle ξ^k over B is isomorphic to $f^*(\gamma_k^\infty)$ for some continuous map f: $B \longrightarrow G_k(F^\infty)$ as vector bundles over F (p. 31 of [2]). If we put $[B, G_k(F^\infty)]$ = the set of homotopy classes of maps from B to $G_k(F^\infty)$, then there exists an one-to-one correspondence between $[B, G_k(F^\infty)]$ and $\mathrm{Vect}_k(B)$, This function ϕ_B: $[B, G_k(F^\infty)] \longrightarrow \mathrm{Vect}_k(B)$ is defined by $\phi_B([f]) = \{f^*(\gamma_k^\infty)$ for $[f] \in [B, G_k(F^\infty)]$. Since $[\text{——}, G_k(F^\infty)]$: $\mathbf{Pa} \longrightarrow \mathbf{Ens}$ is a cofunctor, in fact ϕ is a natural transformation between cofunctors $[\text{——}, G_k(F^\infty)]$ and Vect_k, where ϕ is the family of functions ϕ_B for $B \in \mathbf{Pa}$.

PROPOSITION 2. *ϕ is a natural equivalence from* $[\text{——}, G_k(F^\infty)]$ *to* Vect_k.

PROOF. At first, to prove that ϕ is a natural transformation, we take a homotopy class $[f]: B_1 \longrightarrow B$ of maps in **Pa**. Then we have the commutative diagram

$$\begin{array}{ccc} [B,\ G_k(F^\infty)] & \xrightarrow{\phi_B} & \mathrm{Vect}_k(B) \\ {\scriptstyle [[f],\ G_k(F^\infty)]}\downarrow & \copyright & \downarrow{\scriptstyle \mathrm{Vect}_k([f])} \\ [B_1,\ G_k(F^\infty)] & \xrightarrow{\phi_{B_1}} & \mathrm{Vect}_k(B_1) \end{array}$$

that is, for $[g] \in [B,\ G_k(F^\infty)]$

$$\mathrm{Vect}_k([f])\,(\phi_B([g])) = \mathrm{Vect}_k([f])\,(\{g^*(\gamma_k^\infty)\}) = \{f^* \circ g^*(\gamma_k^\infty)\}$$

$$\phi_{B_1}([[f],\ G_k(F^\infty)])([g]) = \phi_{B_1}([g] \circ [f]) = \phi_{B_1}([g \circ f]) = \{(g \circ f)^*(\gamma_k^\infty)\}$$

$$= \{f^* \circ g^*(\gamma_k^\infty)\}.$$

For each $B \in \mathrm{Pa}$, ϕ_B is surjective and injective ([2]), and therefore ϕ is a natural equivalence.

2. Cofunctor $S_F(X, A)$.

Suppose a pair (X, A) of spaces. Two difference isomorphisms over (X, A), α: $\xi_0 \longrightarrow \xi_1$ and β: $\eta_0 \longrightarrow \eta_1$ are isomorphic if there exist isomorphisms u_i: $\xi_i \longrightarrow \eta_i$ (over X) for $i=0, 1$ such that the following diagram of isomorphisms is commutative.

$$\begin{array}{ccc} \xi_0 | A & \xrightarrow{\alpha} & \xi_1 | A \\ u_0 \downarrow & \copyright & \downarrow u_1 \\ \eta_0 | A & \xrightarrow{\beta} & \eta_1 | A \end{array}$$

Let $S_k(X,A)$ be the set of all difference isomorphism classes of k-dimensional, F-vector bundles over (X,A). For a continuous function $f\colon (X,A)\longrightarrow(Y,B)$ with $f(A)\subset B$, we define $S_k(f)\colon S_k(Y,B)\longrightarrow S_k(X,A)$ by $[S_k(f)](\{\xi\})=\{f^*(\xi)\}$, where $\{\xi\}\in S_k(Y,B)$ and $\{f^*(\xi)\}\in S_k(X,A)$.

PROPOSITION 3. *If $f\colon (X,A)\longrightarrow(Y,B)$ is a continuous map such that $f(A)\subset B$, and if $\beta\colon \eta_0\longrightarrow\eta_1$ is a difference isomorphism over (Y,B), then $f^*(\beta)\colon f^*(\eta_0)\longrightarrow f^*(\eta_1)$ is a difference isomorphism over (X,A).*

PROOF. We want to show that $f^*(\beta)\colon f^*(\eta_0)|A\longrightarrow f^*(\eta_1)|A$ is an isomorphism. For the inclusion map $i\colon A\longrightarrow X$, $(f\circ i)^*(\eta_0)=i^*\circ f^*(\eta_0)\cong i^*\circ f^*(\eta_0|B)\cong f^*(\eta_0)|A$ and $(f\circ i)^*(\eta_1)=i^*\circ f^*(\eta_1)\cong i^*\circ f^*(\eta_1|B)\cong f^*(\eta_1)|A$, and therefore $f^*(\beta)\colon f^*(\eta_0)|A\longrightarrow f^*(\eta_1)|A$ is an isomorphism, because of $(f\circ i)^*(\beta)\colon (f\circ i)^*(\eta_0|B)\longrightarrow(f\circ i)^*(\eta_1|B)$ is an isomorphism induced from the isomorphism $\beta\colon \eta_0|B\longrightarrow\eta_1|B$.

By this proposition we see that $S_k(f)$ is well defined. Let $\mathbf{C}\times\mathbf{C}_0$ be the category of all pairs of topological spaces and maps between pairs. Then

$$S_k\colon \mathbf{C}\times\mathbf{C}_0 \longrightarrow \mathbf{Ens}$$

is a cofunctor. Put $S_F(X,A)=\{S_k(X,A)\}_{k=0,1,\cdots,}$. Then

$$S_F\colon \mathbf{C}\times\mathbf{C}_0 \longrightarrow \mathbf{Ens}$$

is a cofunctor. We define a commutative semigroup structure on $S_F(X,A)$, using the quotient function of the Whitney sum operation defined as usual by $\alpha\oplus\beta\colon \xi_0\oplus\eta_0\longrightarrow\xi_1\oplus\eta_1$ for $\alpha\colon \xi_0\longrightarrow\xi_1$ and $\beta\colon \eta_0\longrightarrow\eta_1$. Of course, if $\alpha\colon \xi_0\longrightarrow\xi_1$ and $\beta\colon \eta_0\longrightarrow\eta_1$ are difference isomorphisms over (X,A), then $\alpha\oplus\beta\colon \xi_0\oplus\eta_0\longrightarrow\xi_1\oplus\eta_1$ is a difference isomorphism over (X,A). Let $\mathbf{Sg}$ be the category of all semigroups and semigroup maps. Then

$$S_F\colon \mathbf{C}\times\mathbf{C}_0 \longrightarrow \mathbf{Sg}$$

is a cofunctor.

3. The main theorem.

In this section, we assume that A be a subcomplex or subspace of a finite CW-complex X.

THEOREM 2. (Main theorem) *For (X,A), $S_F(X,A)$ is a sub-semigroup of* $\mathrm{Vect}_F(A)$. *If X is deformable into a subspace A, then $S_F(X,A)\cong\mathrm{Vect}_F(A)$ as* *semigroups.*

To prove this theorem we need the following lemmas.

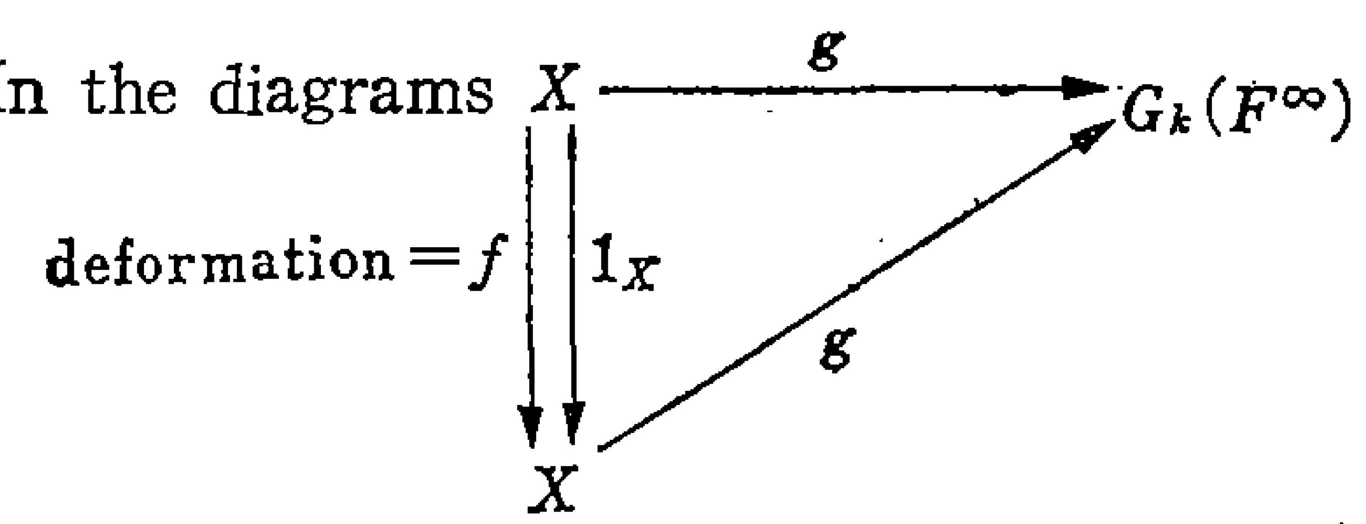
LEMMA 1. *Let ξ_0 and ξ_1 be two vector bundles over X,* If u: $\xi_0|A \longrightarrow \xi_1|A$ *is a vector bundle morphism, then there exists a unique vector bundle morphism v: $\xi_0 \longrightarrow \xi_1$ such that $v|A=u$, where A is a subcomplex.*

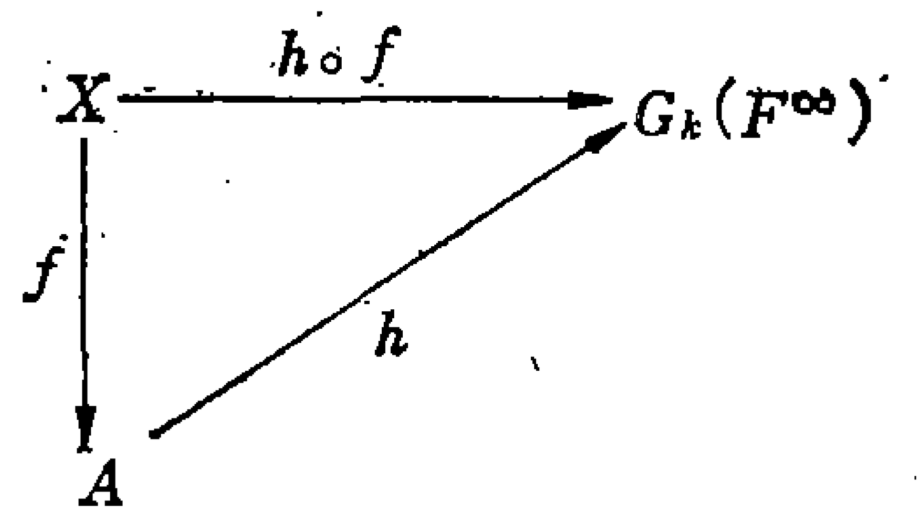
PROOF. Define $E(\mathrm{Hom}_F(\xi_0, \xi_1)) = \bigcup_{x\in X} \mathrm{Hom}_F((\xi_0)_x, (\xi_1)_x)$, where $(\xi_0)_x$ is the fibre at $x\in X$ of ξ_0, and so on. Since each $\mathrm{Hom}_F((\xi_0)_x, (\xi_1)_x)$ for all $x\in X$ is a vector space, $\mathrm{Hom}_F(\xi_0, \xi_1)=(E(\mathrm{Hom}_F(\xi_0, \xi_1)), P, X)$ is a vector bundle over X, where $P^{-1}(x) = \mathrm{Hom}_F((\xi_0)_x, (\xi_1)_x)$. Then we can view u as a cross section of $\mathrm{Hom}_F(\xi_0, \xi_1)$ over A, i.e., for each $x\in A$, $u(x)=u|(\xi_0)_x$: $(\xi_0)_x \longrightarrow (\xi_1)_x$. Since every vector space is contractible, the fibre of $\mathrm{Hom}_F(\xi_0, \xi_1)$ is also contractible. Therefore, by Theorem 1, u is extended to a unique cross section v: $X \longrightarrow \mathrm{Hom}_F(\xi_0, \xi_1)$. In this case, $v=\{v(x): (\xi_0)_x \longrightarrow (\xi_1)x \mid x\in X\}$ is a vector bundle morphism v: $\xi_0 \longrightarrow \xi_1$ which prolongs u.

LEMMA 2. *If X is deformable into A, then* $\mathrm{Vect}_F(X)\cong\mathrm{Vect}_F(A)$ *as semigroups.*

PROOF. Since X is deformable into A, there is a continuous map f: $X \longrightarrow A$ such that $i\circ f\simeq 1_X$ (homotopic), where i: $A \longrightarrow X$ is the inclusion map ([4]). Recall that there is an one-to-one correspondence between $[B, G_k(F^\infty)]$ and $\mathrm{Vect}_F(B)$ for $B\in \mathrm{Pa}$ and for all $k=0, 1, \cdots$.

In the diagrams





We see that $g\simeq g\circ f=g|A\circ f$ and $h\simeq h\circ f|A$, because of $f\simeq 1_X$ implies that $f|A\simeq 1_A$. Thus there is the one-to-one correspondence

$$\phi:\ [X, G_k(F^\infty)] \longrightarrow [A, G_k(F^\infty)]$$

defined by $\phi([g])=[g|A]$. The inverse ϕ^{-1} of ϕ is defined by $\phi^{-1}([h])=[h\circ f]$. Then $\phi^{-1}\circ\phi([g])=\phi^{-1}([g|A])=[g|A\circ f]=[g]$ and $\phi\circ\phi^{-1}([h])=\phi([h\circ f])=[h\circ f|A]=[h]$

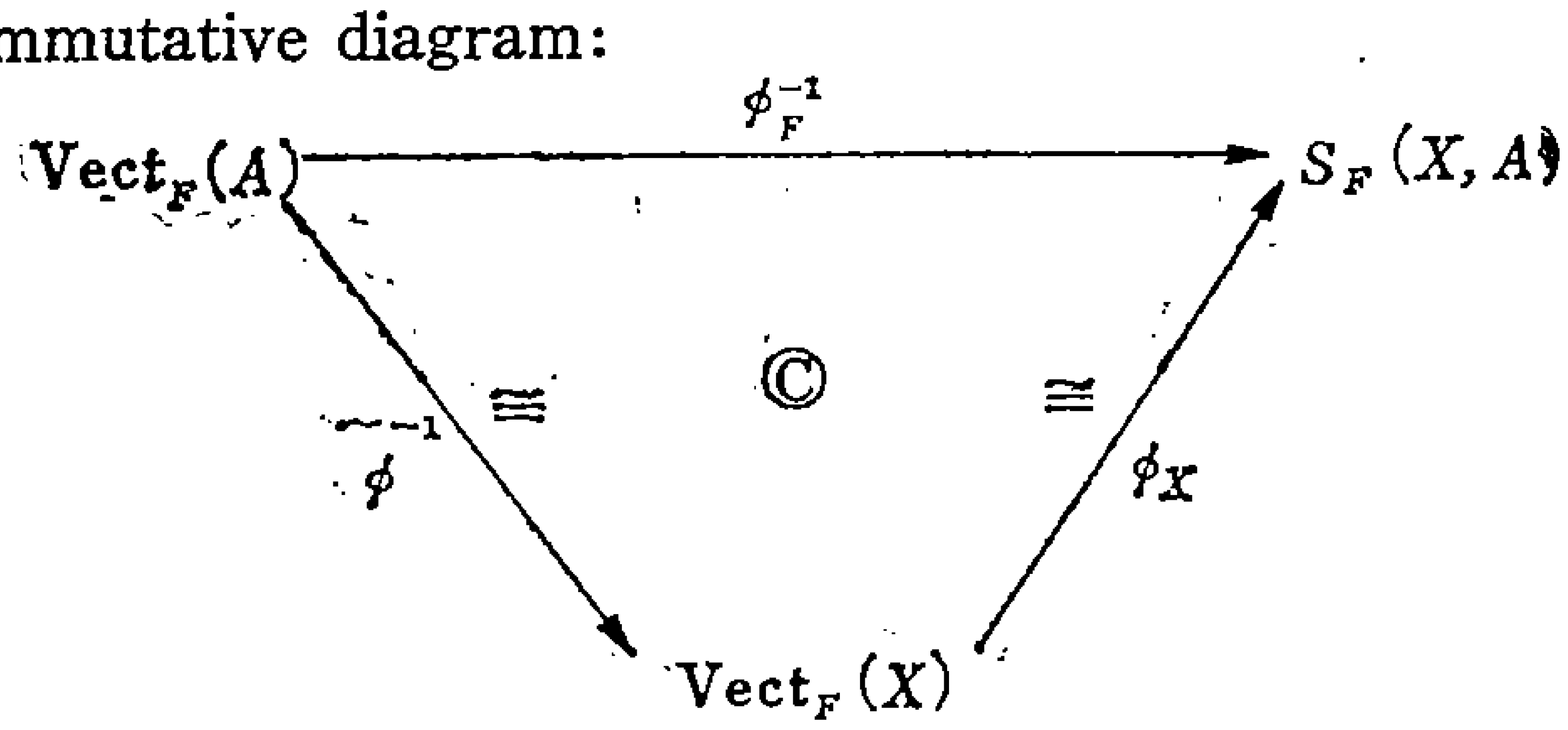
Therefore we define $\tilde{\phi}$: $\mathrm{Vect}_F(X) \longrightarrow \mathrm{Vect}_F(A)$ by $\tilde{\phi}(\{g^*(\gamma_k^\infty)\})=\{(g|A)^*(\gamma_k^\infty)\}$. Then $\tilde{\phi}$ is a semigroup isomorphism.

PROOF of Theorem 2. Let ξ_0 and ξ_1 be two vector bundles over X. By Lemma 1, if there is a vector bundle isomorphism $\xi_0|A \longrightarrow \xi_1|A$ over A then $\{\xi_0\} = \{\xi_1\}$ in $S_F(X,A)$. Therefore the morphism

$$\phi_F:\ S_F(X,A) \longrightarrow \mathrm{Vect}_F(A)$$

defined by $\phi_F(\{\xi\}) = \{\xi|A\}$ is injective and preserves Whitney sum. (Note that the class $\{\xi\}$ in $\phi_F(\{\xi\})$ is the difference isomorphism class containing ξ and the $\{\xi|A\}$ is the isomorphism class containing $\xi|A$). Thus ϕ_F is a monomorphism between semigroups, and therefore $S_F(X,A)$ is isomorphic to a sub-semigroup of $\mathrm{Vect}_F(A)$.

Let us assume that X is deformable into A. Then there is a continuous map f: $X \longrightarrow A \subset X$ such that $f \simeq 1_X$. Lemma 2 says that $\xi_0 \cong \xi_1$ over X iff $\xi_0|A \cong \xi_1|A$ over A in our situation. That is, there is a semigroup isomorphism ϕ_X: $\mathrm{Vect}_F(X) \cong S_F(X,A)$ defined by $\phi_X(\{\xi\}) = \{\xi\}$. Define ϕ_F^{-1}: $\mathrm{Vect}_F(A) \longrightarrow S_F(X, A)$ by the commutative diagram:



Then $\phi_F^{-1} \circ \phi_F = 1_{S_F(X,A)}$ and $\phi_F \circ \phi_F^{-1} = 1_{\mathrm{vect}_F(A)}$.

Let us denote the *completions* of $\mathrm{Vect}_F(A)$ and $S_F(X, A)$ by $K_F(A)$ and $KS_F(X,A)$, respectively ([2]). Then, from Theorem 2 we easily obtain the following.

COROLLARY 1. *$KS_F(X,A)$ is a subgroup of $K_F(A)$. If X is deformable into A, then $KS_F(X,A) \cong K_F(A)$ as abelian groups.*

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